

# Expanded Vandermonde powers and sum rules for the two-dimensional one-component plasma

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## Abstract

The two-dimensional one-component plasma (2dOCP) is a system of  $N$  mobile particles of the same charge  $q$  on a surface with a neutralising background. The Boltzmann factor of the 2dOCP at temperature  $T$  can be expressed as a Vandermonde determinant to the power  $\Gamma = q^2/(k_B T)$ . Recent advances in the theory of symmetric and anti-symmetric Jack polynomials provide an efficient way to expand this power of the Vandermonde in their monomial basis, allowing the computation of several thermodynamic and structural properties of the 2dOCP for  $N$  values up to 14 and  $\Gamma$  equal to 4, 6 and 8. In this work, we explore two applications of this formalism to study the moments of the pair correlation function of the 2dOCP on a sphere, and the distribution of radial linear statistics of the 2dOCP in the plane.

Key words: Coulomb gas; one-component plasma; Jack polynomials; sum rules.

## 1 Introduction

Generally the two-dimensional one-component plasma (2dOCP) refers to a model in classical statistical mechanics consisting of  $N$  mobile point particles of the same charge  $q$  and a smeared out neutralising background, confined in a two-dimensional surface. All the charges (point and continuous) interact via the solution  $v(\vec{r}, \vec{r}')$  of the Poisson equation on that surface. For the plane this is

$$v(\vec{r}, \vec{r}') = -\log(|\vec{r} - \vec{r}'|/L), \quad (1.1)$$

where  $L$  is an arbitrary length scale, which will be set equal to one from now on, while for a sphere of radius  $R$ ,

$$v((\theta, \phi), (\theta', \phi')) = -\log(2R|u'v - uv'|), \quad (1.2)$$

where with  $(\theta, \phi)$  and  $(\theta', \phi')$  spherical coordinates,  $u, v$  (and similarly  $u', v'$ ) are the Cayley-Klein parameters

$$u := \cos(\theta/2)e^{i\phi/2}, \quad v := -i \sin(\theta/2)e^{-i\phi/2} \quad (1.3)$$

(see e.g. [9, §15.6.1]). In the latter case, due to the rotational invariance of the sphere, the background-particle interaction only contributes a constant, and the Boltzmann factor, at temperature  $T$ , is then computed to be

$$\left(\frac{1}{2R}\right)^{N\Gamma/2} e^{\Gamma N^2/4} \prod_{1 \leq j < k \leq N} |u_k v_j - u_j v_k|^\Gamma, \quad (1.4)$$

where  $\Gamma = q^2/(k_B T)$ . In the former case, the particles couple to the background via a harmonic potential towards the origin, and the Boltzmann factor is then (see e.g. [9, eq. (1.72)])

$$A_\Gamma e^{-\pi\Gamma\rho_b \sum_{j=1}^N |\vec{r}_j|^2/2} \prod_{1 \leq j < k \leq N} |\vec{r}_k - \vec{r}_j|^\Gamma, \quad A_\Gamma = e^{-\Gamma N^2((1/2)\log R - 3/8)}, \quad (1.5)$$

where  $\rho_b$  is the density of the background. Although the derivation of (1.5) requires that the mobile particles and background be confined to a disk of radius  $R$ , for our purposes below we will relax this constraint by allowing the particles to locate anywhere in the plane. This situation will be called the soft disk geometry.

Both (1.5) and (1.4) have significance in a field of theoretical physics quite distinct from that of classical plasmas. Consider for definiteness (1.5). According to the well known Vandermonde determinant identity (see e.g. [9, eq. (1.173)]), for  $\Gamma = 2$  and after introducing polar coordinates  $\vec{r} = (r, \theta)$  and then complex coordinates  $z_j = r_j e^{i\theta_j}$ , this can be written in the Slater determinant form

$$A_2 \left| \det[\phi_{j-1}(z_k)]_{j,k=1,\dots,N} \right|^2, \quad \phi_{j-1}(z) = z^{j-1} e^{-\pi\rho_b r^2} \quad (1.6)$$

Moreover, the single particle wave functions  $\phi_{j-1}(z)$ ,  $j = 1, \dots, N$  are, with  $\pi\rho_b = 1/4l^2$   $l = \sqrt{\hbar c/eB}$  the magnetic length, ground state (lowest Landau level) eigenfunctions of the Schrödinger equation for particles confined to the plane in a perpendicular magnetic field of strength  $B$  (see e.g. [9, §15.2.1]). The integer  $j$  has the interpretation of being proportional to the centre of the corresponding classical cyclotron orbits. Thus (1.6) is the absolute value squared of  $N$  non-interacting fermions in the plane, subject to a perpendicular magnetic field which in turn is required to be sufficiently strong that the spin degree of freedom of the fermions is frozen out. For  $\Gamma/2$  an odd integer, it is a celebrated result of Laughlin [15], that the (un-normalized) wave function

$$\left( \det[\phi_{j-1}(z_k)]_{j,k=1,\dots,N} \right)^{\Gamma/2} = e^{-\pi\rho_b \Gamma \sum_{j=1}^N |\vec{r}_j|^2/2} \prod_{1 \leq j < k \leq N} (z_k - z_j)^{\Gamma/2}$$

is an accurate trial wave function for the fractional quantum Hall effect in the case of filling fraction  $2/\Gamma$ . Furthermore, the same holds for

$$\prod_{1 \leq j < k \leq N} (u_k v_j - u_j v_k)^{\Gamma/2}$$

in relation to fractional quantum Hall states on the sphere [18].

We have highlighted the plasma system in the plane, and on a sphere. As is usual in statistical mechanics, bulk properties such as the free energy per particle are expected be

independent of the boundary conditions. In two-dimensions, the sphere is distinguished by the usual correction to the bulk free energy per particle — the surface tension — being absent due to the homogeneity of the geometry. This feature of the sphere makes it well suited for studies in which the aim is to extrapolate bulk values from finite  $N$  data [6, 10, 19]. Extrapolation of finite  $N$  data for the 2dOCP is to be a primary concern of this present paper, and as such we too will make use of spherical geometry and consider the plasma system specified by the Boltzmann factor (1.4).

Specifically, we aim to probe the finite  $N$  analogues of the known sum rules for the bulk two-particle correlation function  $\rho_{(2)}(\vec{r}, \vec{0})$ , or more conveniently the truncated bulk two-particle correlation function  $\rho_{(2)}^T(\vec{r}, \vec{0}) := \rho_{(2)}(\vec{r}, \vec{0}) - \rho^2$ , where  $\rho$  denotes the particle density. These sum rules read [17, 13]

$$\frac{1}{\rho} \int_{\mathbb{R}^2} \rho_{(2)}^T(\vec{r}, \vec{0}) d\vec{r} = -1 \quad (1.7)$$

$$\int_{\mathbb{R}^2} r^2 \rho_{(2)}^T(\vec{r}, \vec{0}) d\vec{r} = -\frac{2}{\pi\Gamma} \quad (1.8)$$

$$\rho \int_{\mathbb{R}^2} r^4 \rho_{(2)}^T(\vec{r}, \vec{0}) d\vec{r} = -\frac{16}{(\pi\Gamma)^2} \left(1 - \frac{\Gamma}{4}\right) \quad (1.9)$$

$$\rho^2 \int_{\mathbb{R}^2} r^6 \rho_{(2)}^T(\vec{r}, \vec{0}) d\vec{r} = -\frac{18}{(\pi\Gamma)^3} \left(\Gamma - 6\right) \left(\Gamma - \frac{8}{3}\right). \quad (1.10)$$

It is straightforward to show from the definitions that (1.7) is exact for all  $N$  on the sphere. In fact the sphere plays no specific role; replacing  $\rho$  on the LHS by  $\rho(\vec{0})$  this holds for a general one-component system. Less obvious and special to the sphere geometry is that (1.8), appropriately modified, can also be made exact for finite  $N$ . The derivation is given in Section 3 below.

To probe (1.9) and (1.10) for finite  $N$  we have available an exact expression for  $\rho_{(2)}^T(\vec{r}, \vec{r}')$  at the special coupling  $\Gamma = 2$  [5]. More generally, for  $\Gamma$  an even integer (1.4) is the absolute value squared of a multivariable polynomial. In the case that  $\Gamma/2$  is also even, a recent advance [3] identifies the polynomial in terms of the symmetric Jack polynomials (see [14] and [9] for textbook treatments), while for  $\Gamma/2$  odd an even more recent paper [4] shows that the polynomial is an anti-symmetric Jack polynomial. These Jack polynomials have certain structural properties (related to their monomial basis expansion) which makes them the most efficient known way to carry out exact numerical computations at these couplings. We review the formalism of such computation methods in Section 2, and we give too an alternative derivation to the result of [4], [21] for the eigenoperator of anti-symmetric Jack polynomials. The results of calculations based on this formalism are reported in Section 3. In Section 4, an approximation to the moments is probed using a formalism based on the direct correlation function.

One viewpoint on the moments of  $\rho_{(2)}^T(\vec{r}, \vec{0})$  is that they are the averaged value of the linear statistic  $\sum_{l=1}^N |\vec{r}_l|^{2n}$  in the system perturbed by a particle being fixed at the origin, minus the average value of this same linear statistic before the perturbation. This suggests the question of computing fluctuation formulas for linear statistics, and we take up this task in Section 5. Section 5 is independent from Sections 3 and 4. Some concluding remarks are made in Section 6.

## 2 Preliminary material

In this section the expansion formulas underlying the exact calculations of the moments — both analytic and numeric — will be presented. Before doing so, for the sphere geometry, it is convenient to map the particles to the plane by applying a stereographic projection. With the latter carried out by mapping from the south pole to the plane tangent to the north pole, and with  $(\theta, \phi)$  the usual spherical coordinates, this is specified by the equation

$$z = 2Re^{i\phi} \tan \frac{\theta}{2}, \quad z = x + iy,$$

and we then have

$$\begin{aligned} & \left(\frac{1}{2R}\right)^{N\Gamma/2} e^{\Gamma N^2/2} \prod_{1 \leq j < k \leq N} |u_k v_j - u_j v_k|^\Gamma dS_1 \cdots dS_N \\ &= \left(\frac{1}{2R}\right)^{N\Gamma/2} e^{\Gamma N^2/2} \prod_{j=1}^N \frac{1}{(1 + |z_j|^2/(4R^2))^{2+\Gamma(N-1)/2}} \prod_{1 \leq j < k \leq N} \left|\frac{z_j - z_k}{2R}\right|^\Gamma d\vec{r}_1 \cdots d\vec{r}_N. \end{aligned} \quad (2.1)$$

As previously remarked, for  $\Gamma$  even the product of differences in both (1.5) and (1.4) is the absolute value squared of a polynomial. This remains true of the stereographic projection of (1.4) as seen in (2.1). In the case  $\Gamma = 4p$  the polynomial is symmetric, while in the case  $\Gamma = 4p+2$  the polynomial is anti-symmetric. As a consequence, the two case need to be treated separately.

### 2.1 The case $\Gamma = 4p$

Following [19], let  $\mu = (\mu_1, \dots, \mu_N)$  be a partition of  $pN(N-1)$  such that

$$2p(N-1) \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_N \geq 0, \quad (2.2)$$

and, with  $m_i$  denoting the corresponding frequency of the integer  $i$  in the partition, define the corresponding monomial symmetric function by

$$m_\mu(z_1, \dots, z_N) = \frac{1}{\prod_i m_i!} \sum_{\sigma \in S_N} z_{\sigma(1)}^{\mu_1} \cdots z_{\sigma(N)}^{\mu_N}.$$

We expand

$$\prod_{1 \leq j < k \leq N} (z_k - z_j)^{2p} = \sum_{\mu} c_{\mu}^{(N)}(2p) m_{\mu}(z_1, \dots, z_N). \quad (2.3)$$

The significance of knowledge of  $\{c_{\mu}^{(N)}(2p)\}$  in this expansion is that a simple calculation shows [19, eq. (2.7)]

$$\begin{aligned} I_{N,\Gamma}[g] &:= \int_{\mathbb{R}^2} d\vec{r}_1 g(r_1^2) \cdots \int_{\mathbb{R}^2} d\vec{r}_N g(r_N^2) \prod_{1 \leq j < k \leq N} |\vec{r}_k - \vec{r}_j|^\Gamma \\ &= N! \pi^N \sum_{\mu} \frac{(c_{\mu}^{(N)}(2p))^2}{\prod_i m_i!} \prod_{l=1}^N G_{\mu_l}, \end{aligned} \quad (2.4)$$

where

$$G_{\mu_l}[g] := 2 \int_0^\infty r^{1+2\mu_l} g(r^2) dr. \quad (2.5)$$

## 2.2 The case $\Gamma = 4p + 2$

We again follow [19]. We now take  $\mu$  to be a partition of  $(p+1)N(N-1)$  such that

$$(2p+1)(N-1) \geq \mu_1 > \mu_2 > \cdots > \mu_N \geq 0 \quad (2.6)$$

(note that the strict inequalities between the parts of the partition implies  $m_i = 1$ , in the notation used below (2.2)). We then expand

$$\prod_{1 \leq j < k \leq N} (z_k - z_j)^{2p+1} = \sum_{\mu} c_{\mu}^{(N)}(2p+1) \mathcal{A}(z_1^{\mu_1} \cdots z_N^{\mu_N}), \quad (2.7)$$

where  $\mathcal{A}$  denotes anti-symmetrization. From the definition of the Schur polynomials  $s_{\mu}(z_1, \dots, z_N)$ , and with  $\delta_N := (N-1, N-2, \dots, 0)$ , this is equivalent to the expansion

$$\prod_{1 \leq j < k \leq N} (z_k - z_j)^{2p} = \sum_{\mu} c_{\mu}^{(N)}(2p+1) s_{\mu-\delta_N}(z_1, \dots, z_N), \quad (2.8)$$

and now for the integral (2.4) we have

$$I_{N,\Gamma}[g] = N! \pi^N \sum_{\mu} (c_{\mu}^{(N)}(2p+1))^2 \prod_{l=1}^N G_{\mu_l}. \quad (2.9)$$

## 2.3 The coefficients $c_{\mu}^{(N)}(2p)$ and $c_{\mu}^{(N)}(2p+1)$

It has recently been observed [3, 4] that the products in (2.3) and (2.8) can be expressed in terms of the Jack symmetric polynomial  $P_{\kappa}^{(\alpha)}(z)$  and Jack anti-symmetric polynomial  $S_{\kappa}^{(\alpha)}(z)$  respectively, where  $z := (z_1, \dots, z_N)$ . Generally the nonsymmetric Jack polynomials are eigenfunctions of the differential operator [9, eq. (11.63)]

$$\begin{aligned} \tilde{H}^{(C, Ex)} &= \sum_{j=1}^N \left( z_j \frac{\partial}{\partial z_j} \right)^2 + \frac{N-1}{\alpha} \sum_{j=1}^N z_j \frac{\partial}{\partial z_j} \\ &\quad + \frac{2}{\alpha} \sum_{1 \leq j < k \leq N} \frac{z_j z_k}{z_j - z_k} \left( \left( \frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_k} \right) - \frac{1 - M_{jk}}{z_j - z_k} \right) \end{aligned} \quad (2.10)$$

where  $M_{ij}$  interchanges  $z_i$  and  $z_j$ . The operators of symmetrization and anti-symmetrization commute with this operator, and so with  $M_{jk} = 1$  and  $M_{jk} = -1$  respectively, (2.10) is the differential eigenoperator for  $P_{\kappa}^{(\alpha)}(z)$  and  $S_{\kappa}^{(\alpha)}(z)$ . This characterisation specifies the polynomials uniquely when supplemented by the structural formulas

$$P_{\kappa}^{(\alpha)}(z) = m_{\kappa}(z) + \sum_{\rho < \kappa} c_{\kappa\rho}(\alpha) m_{\rho}(z) \quad (2.11)$$

$$S_{\kappa}^{(\alpha)}(z) = s_{\kappa-\delta_N}(z) + \sum_{\rho-\delta_N < \kappa-\delta_N} \tilde{c}_{\kappa\rho}(\alpha) s_{\rho-\delta_N}(z), \quad (2.12)$$

where in (2.12) all parts of  $\kappa$  are required to be distinct, and  $\mu < \kappa$  refers to the dominance partial ordering on partitions  $|\mu| < |\kappa|$ , specified by  $\sum_{i=1}^k \mu_i \leq \sum_{i=1}^k \kappa_i$ , ( $k = 1, \dots, N$ ).

We remark that in the work [4] the anti-symmetric polynomials, defined as in [1] as the anti-symmetric Jack polynomials, were not considered directly. Rather attention was focussed on  $\prod_{1 \leq i < j \leq N} (z_i - z_j) P_\kappa^{(\alpha)}(z)$ , which was shown to be the polynomial eigenfunction of the differential operator

$$\sum_{j=1}^N \left( z_j \frac{\partial}{\partial z_j} \right)^2 + \frac{1}{2} \left( \frac{1}{\alpha} - 1 \right) \sum_{i \neq j} \left( \frac{z_i + z_j}{z_i - z_j} \left( z_i \frac{\partial}{\partial z_j} - z_j \frac{\partial}{\partial z_i} \right) - 2 \frac{z_i^2 + z_j^2}{(z_i - z_j)^2} \right).$$

Simple manipulation reduces this to

$$\begin{aligned} & \sum_{j=1}^N \left( z_j \frac{\partial}{\partial z_j} \right)^2 + \left( \frac{1}{\alpha} - 1 \right) \sum_{i \neq j} \left( \frac{z_i z_j}{z_i - z_j} \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) - 2 \frac{z_i z_j}{(z_i - z_j)^2} \right) \\ & + \left( \frac{1}{\alpha} - 1 \right) (N-1) \sum_{j=1}^N z_j \frac{\partial}{\partial z_j} - \left( \frac{1}{\alpha} - 1 \right) N(N-1). \end{aligned} \quad (2.13)$$

Comparison with (2.10) reveals that up to a constant this is equal to  $\tilde{H}^{(C, Ex)}$  with  $\alpha$  replaced by  $\alpha/(1-\alpha)$  and  $M_{ij} = -1$ , telling us that (2.13) is an eigenoperator for  $S_\kappa^{(\alpha/(1-\alpha))}(z)$ . This conclusion is consistent with the known relation [1]

$$S_\kappa^{(\alpha/(1-\alpha))}(z) = \prod_{1 \leq i < j \leq N} (z_i - z_j) P_\kappa^{(\alpha)}(z).$$

The products in (2.3) and (2.8) relate to the Jack polynomials with a negative parameter. Explicitly we have [3] (see [2] for a derivation based on earlier literature)

$$\prod_{1 \leq j < k \leq N} (z_k - z_j)^{2p} = P_{2p\delta_N}(z; -2/(2p-1)) \quad (2.14)$$

and [4]

$$\prod_{1 \leq j < k \leq N} (z_k - z_j)^{2p+1} = P_{(2p+1)\delta_N}(z; -2/(2p+1)), \quad (2.15)$$

where generally for  $c \in \mathbb{Z}^+$ ,  $c\kappa := (c\kappa_1, c\kappa_2, \dots, c\kappa_N)$ . The computational advantage of these formulas in computing the coefficients  $c_\mu^{(N)}$  (2.3) and (2.8) is that the partitions  $\mu$  for which  $c_\mu^{(N)}$  are nonzero is greatly restricted by the dominance ordering exhibited in (2.11) and (2.12). Furthermore the eigenfunction characterization allows recursive formulas for the coefficients in these formulas, and thus the coefficients  $\{c_\mu^{(N)}\}$  to be obtained.

In relation to (2.11), let  $\rho = (\rho_1, \rho_2, \dots, \rho_N)$ , and let  $\mu$  be constructed from  $\rho$  by first adding  $r$  to  $\rho_i$  and subtracting  $r$  from  $\rho_j$  with the requirement that upon suitable reordering to be a partition it satisfies  $\rho < \mu \leq \kappa$ . Also define

$$e_\kappa(\alpha) := \sum_{i=1}^N \kappa_i \left( \kappa_i - 1 - \frac{2}{\alpha} (i-1) \right).$$

Then [16, §VI.4 Example 3(d)]

$$c_{\kappa\rho} = \frac{1}{e_{\kappa}(\alpha) - e_{\rho}(\alpha)} \frac{2}{\alpha} \sum_{\rho < \mu \leq \kappa} ((\rho_i + r) - (\rho_j - r)) c_{\kappa\mu}(\alpha). \quad (2.16)$$

Regarding now (2.12), let  $\rho$  and  $\mu$  be as above and define

$$e_{\kappa}^F(\alpha) := \sum_{i=1}^N \kappa_i (\kappa_i + 2i(1 - \frac{1}{\alpha})).$$

Further define  $(-1)^{N_{\text{sw}}}$  as the sign of the permutation needed to be applied to  $\mu$  so that it is a partition. Then we have [4, 21]

$$\tilde{c}_{\kappa\rho} = \frac{1}{e_{\kappa}^F(\alpha) - e_{\rho}^F(\alpha)} \frac{2}{\alpha} \sum_{\rho < \mu \leq \kappa} (\rho_i - \rho_j) \tilde{c}_{\kappa\mu}(\alpha) (-1)^{N_{\text{sw}}}. \quad (2.17)$$

### 3 Moments of the correlation function on the sphere

#### 3.1 Second moment

The partition function of the 2dOCP on a sphere is [5, 19] [9, §15.6]

$$Q = \frac{1}{N!} \int e^{-\beta U} dS_1 \dots dS_N \quad (3.1)$$

$$= \frac{e^{\Gamma N^2/4}}{N!(2R)^{N^2\Gamma/2}} \int_{\mathbb{R}^{2N}} \prod_{1 \leq j < k \leq N} |z_k - z_j|^{\Gamma} \prod_{j=1}^N \frac{d^2 \vec{r}_j}{\left(1 + \frac{|z_j|^2}{(2R)^2}\right)^{2+(N-1)\frac{\Gamma}{2}}}, \quad (3.2)$$

where  $U$  is the potential energy of the 2dOCP and  $z_j = r_j e^{i\phi_j}$  are the coordinates of the stereographic projection of a sphere of radius  $R$ . Equivalently, one can factor out the radius, introducing  $\tilde{z}_j = z_j/(2R)$ , to obtain

$$Q = \frac{e^{\Gamma N^2/4}}{N!(2R)^{2N(\frac{\Gamma}{4}-1)}} \int_{\mathbb{R}^{2N}} \prod_{1 \leq j < k \leq N} |\tilde{z}_k - \tilde{z}_j|^{\Gamma} \prod_{j=1}^N \frac{d^2 \vec{r}_j}{(1 + |\tilde{z}_j|^2)^{2+(N-1)\frac{\Gamma}{2}}}. \quad (3.3)$$

Let us set

$$Q_1 = \frac{N}{N!} \int e^{-\beta U(z_1, \dots, z_{N-1}, 0)} dS_1 \dots dS_{N-1}, \quad (3.4)$$

where the particle number  $N$  is fixed at the north pole. By definition, the density at the north pole is

$$\rho_{(1)}(0) = \frac{Q_1}{Q} = \rho_b. \quad (3.5)$$

In the coordinates of the stereographic projection of the sphere of radius  $R$ ,  $Q_1$  can be written as

$$Q_1 = \frac{N e^{\Gamma N^2/4}}{N!(2R)^{N^2\Gamma/2}} \int_{\mathbb{R}^{2(N-1)}} \prod_{1 \leq j < k \leq N-1} |z_k - z_j|^{\Gamma} \prod_{j=1}^{N-1} \frac{|z_j|^{\Gamma} d^2 \vec{r}_j}{\left(1 + \frac{|z_j|^2}{(2R)^2}\right)^{2+(N-1)\frac{\Gamma}{2}}} \quad (3.6)$$

or, using the scaled coordinates  $\tilde{z}$ ,

$$Q_1 = \frac{N e^{\Gamma N^2/4}}{(2R)^{2(N\frac{\Gamma}{4}-N-1)}} \int_{\mathbb{R}^{2(N-1)}} \prod_{1 \leq j < k \leq N-1} |\tilde{z}_k - \tilde{z}_j|^\Gamma \prod_{j=1}^{N-1} \frac{|\tilde{z}_j|^\Gamma d^2 \vec{r}_j}{(1 + |\tilde{z}_j|^2)^{2+(N-1)\frac{\Gamma}{2}}} \quad (3.7)$$

Computing the derivative of  $Q_1$  with respect to  $R$ , using expression (3.6) gives

$$\frac{R}{Q} \frac{\partial Q_1}{\partial R} = -\frac{N^2 \Gamma}{2} \rho_{(1)}(0) + 2 \left( 2 + \frac{\Gamma}{2} (N-1) \right) \int \rho_{(2)}(\theta) \sin^2 \frac{\theta}{2} dS. \quad (3.8)$$

Here the last integral appears from the derivative

$$R \frac{\partial}{\partial R} \left( \frac{1}{1 + \frac{|z_j|^2}{(2R)^2}} \right)^{2+(N-1)\frac{\Gamma}{2}} = 2(2 + (N-1)\frac{\Gamma}{2}) \frac{|z_j|^2 / (2R)^2}{\left( 1 + \frac{|z_j|^2}{(2R)^2} \right)^{3+(N-1)\frac{\Gamma}{2}}} \quad (3.9)$$

$$= 2(2 + (N-1)\frac{\Gamma}{2}) \left( \frac{1}{1 + \frac{|z_j|^2}{(2R)^2}} \right)^{2+(N-1)\frac{\Gamma}{2}} \sin^2 \frac{\theta_j}{2}. \quad (3.10)$$

On the other hand, starting with expression (3.7) gives

$$\frac{R}{Q} \frac{\partial Q_1}{\partial R} = 2 \left( (N-1) - N \frac{\Gamma}{4} \right) \rho_{(1)}(0). \quad (3.11)$$

Equating (3.8) and (3.11), using the fact that  $\rho_{(1)}(0) = \rho_b$ , and subtracting on both sides  $\int \rho_b^2 \sin^2(\theta/2) dS = N \rho_b / 2$ , gives

$$\int \left( 2R \sin \frac{\theta}{2} \right)^2 \rho_{(2)}^T(\theta) dS = -\frac{1}{\pi} \frac{N}{(N-1)\frac{\Gamma}{2} + 2}, \quad (3.12)$$

where  $\rho_{(2)}^T(\theta)$  is the truncated pair correlation function between the north pole and a point located on the sphere at  $(\theta, \phi)$ .

In the planar limit  $R \rightarrow \infty$ ,  $N \rightarrow \infty$ , with  $\rho_b = N/(4\pi R^2)$  finite, the length of the chord  $r = 2R \sin(\theta/2)$  becomes the distance between the two points, and one recovers the Stillinger-Lovett second moment sum rule (1.8).

### 3.2 Higher order moments

Let us define

$$\hat{I}_{2n} = \rho_b \left( \frac{\pi \Gamma \rho_b}{2} \right)^n \int (2R \sin(\theta/2))^{2n} h(\theta) dS, \quad (3.13)$$

where  $h$  is the total correlation function

$$h(\theta) = \rho_{(2)}^T(\theta) / \rho_b^2. \quad (3.14)$$



Using the formalism presented in Section 2 and [19], the density at a given point on the sphere is

$$\rho_{(1)}(\theta) = \frac{\rho_b((N-1)\frac{\Gamma}{2}+1)!}{Z_{\text{sphere}}N(1+x^2)^{(N-1)\Gamma/2}} \sum_{\mu} \frac{(c_{\mu}^{(N)}(\Gamma/2))^2}{\prod_i m_i!} \prod_{l=1}^N \mu_l!((N-1)\frac{\Gamma}{2}-\mu_l)! \sum_{k=1}^N \frac{x^{2\mu_k}}{\mu_k!((N-1)\frac{\Gamma}{2}-\mu_k)!} \quad (3.15)$$

where  $\theta$  is the angle from the north pole of the sphere,  $x = \tan(\theta/2)$ , and

$$Z_{\text{sphere}} = \sum_{\mu} \frac{(c_{\mu}^{(N)}(\Gamma/2))^2}{\prod_i m_i!} \prod_{l=1}^N \mu_l!((N-1)\frac{\Gamma}{2}-\mu_l)!, \quad (3.16)$$

is (up to a multiplicative constant) the partition function of the 2dOCP on the sphere. Since the sphere is homogeneous, the density does not depend on  $x$  and should be equal to  $\rho_b$ . This gives an interesting relation satisfied by the coefficients,  $(c_{\mu}^{(N)}(\Gamma/2))^2$

$$N(1+x)^{(N-1)\Gamma/2} = \frac{((N-1)\frac{\Gamma}{2}+1)!}{Z_{\text{sphere}}} \sum_{\mu} \frac{(c_{\mu}^{(N)}(\Gamma/2))^2}{\prod_i m_i!} \prod_{l=1}^N \mu_l!((N-1)\frac{\Gamma}{2}-\mu_l)! \sum_{k=1}^N \frac{x^{2\mu_k}}{\mu_k!((N-1)\frac{\Gamma}{2}-\mu_k)!} \quad (3.17)$$

Putting  $x = 0$ , then relates a sum of all admissible partitions and those with  $\mu_N = 0$ ,

$$N \sum_{\mu} \frac{(c_{\mu}^{(N)}(\Gamma/2))^2}{\prod_i m_i!} \prod_{l=1}^N \mu_l!((N-1)\frac{\Gamma}{2}-\mu_l)! = ((N-1)\frac{\Gamma}{2}+1)! \sum_{\substack{\mu \\ \mu_N=0}} \frac{(c_{\mu}^{(N)}(\Gamma/2))^2}{\prod_i m_i!} \prod_{l=1}^{N-1} \mu_l!((N-1)\frac{\Gamma}{2}-\mu_l)! \quad (3.18)$$

The two-point correlation function is given by

$$\rho_{(2)}(\theta) = \frac{\rho_b^2[(((N-1)\frac{\Gamma}{2}+1)!)]^2}{Z_{\text{sphere}}N^2(1+x^2)^{(N-1)\Gamma/2}} \sum_{\substack{\mu \\ \mu_N=0}} \frac{(c_{\mu}^{(N)}(\Gamma/2))^2}{\prod_i m_i!} \prod_{l=1}^{N-1} \mu_l!((N-1)\frac{\Gamma}{2}-\mu_l)! \sum_{k=1}^{N-1} \frac{x^{2\mu_k}}{\mu_k!((N-1)\frac{\Gamma}{2}-\mu_k)!}. \quad (3.19)$$

From these expressions, we obtain the  $2n$ -moment of the total correlation function

$$\begin{aligned} \hat{I}_{2n} &= \left(\frac{N\Gamma}{2}\right)^2 \left[ \frac{(((N-1)\frac{\Gamma}{2}+1)!)^2}{N((N-1)\frac{\Gamma}{2}+1+n)!Z_{\text{sphere}}} \right. \\ &\quad \times \sum_{\substack{\mu \\ \mu_N=0}} \frac{(c_{\mu}^{(N)}(\Gamma/2))^2 \prod_{l=1}^{N-1} \mu_l!((N-1)\frac{\Gamma}{2}-\mu_l)!}{\prod_i m_i!} \sum_{k=1}^{N-1} \frac{(\mu_k+n)!}{\mu_k!} - \frac{N}{n+1} \left. \right]. \quad (3.20) \end{aligned}$$

In the case  $n = 1$ , the sum

$$\sum_{k=1}^{N-1} \frac{(\mu_k+n)!}{\mu_k!} = \sum_{k=1}^{N-1} (\mu_k+1) = (N-1) \left( \frac{\Gamma N}{4} + 1 \right), \quad (3.21)$$

simplifies and it is independent of the partition  $\mu$ , which lead to the sum rule discussed in the previous section,

$$\hat{I}_2 = \frac{N\Gamma}{\Gamma - N\Gamma - 4} \quad (3.22)$$

Unfortunately, for  $n \geq 2$ , no such simplification seems possible.

For  $\Gamma = 2$ , only one partition appears in the expansion and the result for any  $n$  is

$$\hat{I}_{2n} = -N^n \frac{n!N!}{(N+n)!}, \quad \Gamma = 2. \quad (3.23)$$

In the thermodynamic limit,  $N \rightarrow \infty$ , the values of  $\hat{I}_4$  and  $\hat{I}_6$  are also known for any  $\Gamma$  [13]

$$\hat{I}_4 = \Gamma - 4, \quad (3.24)$$

$$\hat{I}_6 = \frac{3}{4}(\Gamma - 6)(8 - 3\Gamma). \quad (3.25)$$

For finite  $N$ , we computed the moments numerically from (3.20). Since the maximum value of  $N$  for which the numerical calculations were possible is small, the results for the moments show important finite size corrections with respect to the known values for  $N \rightarrow \infty$ . The exact results for  $n = 1$  (any  $\Gamma$ ), and the one for  $\Gamma = 2$ , suggest that the finite size corrections can be understood as a series expansion in powers of  $1/N$ . Indeed, for  $n = 1$ ,

$$\hat{I}_2 = -1 - \sum_{k=1}^{\infty} (-1)^k \left( \frac{4 - \Gamma}{\Gamma} \right)^k \frac{1}{N^k} \quad (3.26)$$

and for  $\Gamma = 2$ ,

$$\begin{aligned} \hat{I}_{2n} = & -n! \left[ 1 - \frac{n(n+1)}{2N} + \frac{n(n+1)(n+2)(1+3n)}{24N^2} \right. \\ & - \frac{n^2(n+1)^2(n+2)(n+3)}{48N^3} + \frac{n(n+1)(n+2)(n+3)(n+4)(15n^3 + 30n^2 + 5n - 2)}{5760N^4} \\ & \left. + O(1/N^5) \right] \end{aligned} \quad (3.27)$$

Therefore, we decided to fit the numerical data obtained for the moments with a  $1/N$  expansion. The results are shown for  $\Gamma = 4$  in tables 3.1, 3.2, 3.3, for  $\Gamma = 6$  in tables 3.4, 3.5, 3.6, and for  $\Gamma = 8$  in tables 3.7, 3.8, 3.9. For  $\Gamma = 4$ , the coefficient of order 0 (which gives the value of  $\hat{I}_{2n}$  when  $N \rightarrow \infty$ ), shows an acceptable convergence to the expected value for  $\hat{I}_4$  and  $\hat{I}_6$ , see tables 3.1, 3.2. Also, from table 3.3 ( $\hat{I}_8$ ,  $\Gamma = 4$ ), the zero order coefficient seems to converge to a value close to  $-30$ . This allows us to obtain an estimate for the value of 8-th moment, for  $\Gamma = 4$ , in the thermodynamic limit,  $\hat{I}_8 \simeq -30$ . Unfortunately, for  $\Gamma = 6$  and  $\Gamma = 8$ , since the maximum value of  $N$  for which the calculation were possible was smaller and the finite size corrections still very large, we were not able to obtain any estimate for  $\hat{I}_8$ .

$N$	$\hat{I}_4$	$=$	$a$	$+$	$b/N$	$+$	$c/N^2$	$+$	$d/N^3$
			$a$		$b$		$c$		$d$
2	-1.06666666666667								
3	-0.73469387755102								
4	-0.552915766738661								
5	-0.437781621713968		0.076709		-2.81362		1.30721		-0.506965
6	-0.361584090880502		-0.042858		-1.37881		-4.31243		6.66705
7	-0.307439760694233		0.0121923		-2.20457		-0.238709		0.0610148
8	-0.267158562552772		-0.00224307		-1.94473		-1.7833		3.09245
9	-0.236094785664912		-0.00181613		-1.9537		-1.72096		2.94899
10	-0.211435346122364		0.0000393273		-1.99823		-1.36657		2.01384
11	-0.191399568920847		0.000150015		-1.99312		-1.41239		2.15017
12	-0.17480728582518		-0.000241804		-1.99036		-1.43984		2.24104
13	-0.160846001528985		-0.000138037		-1.99379		-1.40227		2.10407
14	-0.148938895568825		-0.0000794746		-1.99589		-1.37703		2.00357
$\infty$	0								

Table 3.1:  $1/N$  expansion coefficients for  $\hat{I}_4$  when  $\Gamma = 4$ .

$N$	$\hat{I}_6$	$=$	$a$	$+$	$b/N$	$+$	$c/N^2$	$+$	$d/N^3$
			$a$		$b$		$c$		$d$
4	5.1605471562275								
5	6.11292630320115								
6	6.56602660174796								
7	6.78048920359037		5.49915		27.6653		-150.595		138.066
8	6.87498575554362		5.58979		26.0337		-140.897		119.031
9	6.9075123159669		5.74333		22.8094		-118.48		67.4414
10	6.90772192712509		5.87714		19.5982		-92.9236		0.00502236
11	6.89103601184945		5.92331		18.3514		-81.7487		-33.2425
12	6.86574608931141		5.94863		17.592		-74.1805		-58.301
13	6.83643498600859		5.96747		16.9702		-67.3594		-83.1735
14	6.80567000027031		5.97905		16.5534		-62.3686		-103.044
$\infty$	6								

Table 3.2:  $1/N$  expansion coefficients for  $\hat{I}_6$  when  $\Gamma = 4$ .

$N$	$\hat{I}_8 =$	$a +$	$b/N +$	$c/N^2 +$	$d/N^3$
		$a$	$b$	$c$	$d$
6	40.3968590167648				
7	33.7964087943161				
8	26.9657510022336				
9	20.6758161117942	-49.3234	709.717	-95.7946	-5595.45
10	15.1585480309705	-38.7061	454.9	1932.12	-10946.6
11	10.414558865846	-33.7202	320.28	3138.71	-14536.5
12	6.3618885422448	-31.5027	253.756	3801.74	-16731.8
13	2.89953248501109	-30.4953	220.513	4166.41	-18061.5
14	-0.0685490396317987	-30.1108	206.671	4332.12	-18721.3

Table 3.3:  $1/N$  expansion coefficients for  $\hat{I}_8$  when  $\Gamma = 4$ .

$N$	$\hat{I}_4 =$	$a +$	$b/N +$	$c/N^2 +$	$d/N^3$
		$a$	$b$	$c$	$d$
4	1.77112299465241				
5	1.92727455514225				
6	1.97167595506035				
7	1.99917180579183	2.90767	-14.9074	84.3562	-171.645
8	2.01319334170647	1.72398	6.39891	-42.2979	76.9278
9	2.02020128573558	1.95459	1.55616	-8.62918	-0.556218
10	2.02414460391682	2.08557	-1.5874	16.3883	-66.5709
11	2.02625080279343	1.98854	1.03254	-7.09409	3.29402
12	2.02719976576841	1.98147	1.2446	-9.2077	10.2923
$\infty$	2				

Table 3.4:  $1/N$  expansion coefficients for  $\hat{I}_4$  when  $\Gamma = 6$ .

$N$	$\hat{I}_6 =$	$a +$	$b/N +$	$c/N^2 +$	$d/N^3$
		$a$	$b$	$c$	$d$
4	29.2857260386672				
5	24.3553395496129				
6	19.3570221116088				
7	15.8157978386411	23.4057	-367.801	3052.6	-5949.24
8	13.2137920826172	-1.67692	83.6859	368.754	-681.887
9	11.2455089383337	-0.844904	66.2135	490.229	-961.446
10	9.75173060605997	4.89476	-71.5384	1586.5	-3854.23
11	8.58562698480205	0.124033	57.2712	431.99	-419.314
12	7.65247243254821	-0.286731	69.5941	309.171	-12.6574
$\infty$	0				

Table 3.5:  $1/N$  expansion coefficients for  $\hat{I}_6$  when  $\Gamma = 6$ .

$N$	$\hat{I}_8 =$	$a +$	$b/N +$	$c/N^2 +$	$d/N^3$
		$a$	$b$	$c$	$d$
4	157.436163836164				
5	-33.5270610707337				
6	-196.111013214468				
7	-293.318966661244				
8	-343.529970425059				
9	-366.832915368496	129.267	-11229.9	71322.9	-93943.9
10	-374.243763793112	161.032	-11992.2	77390	-109954
11	-373.310338904328	-86.8351	-5299.81	17406.2	68510.9
12	-368.298029500803	-183.485	-2400.33	-11492.1	64194

Table 3.6:  $1/N$  expansion coefficients for  $\hat{I}_8$  when  $\Gamma = 6$ .

$N$	$\hat{I}_4 =$	$a +$	$b/N +$	$c/N^2 +$	$d/N^3$
		$a$	$b$	$c$	$d$
3	5.9504132231405				
4	5.35216744227873				
5	5.31949584221798				
6	5.03608123946663	-6.67707	150.385	-623.767	858.777
7	4.90673278031246	11.9835	-129.523	757.114	-1380.49
$\infty$	4				

Table 3.7:  $1/N$  expansion coefficients for  $\hat{I}_4$  when  $\Gamma = 8$ .

$N$	$\hat{I}_6$	$=$	$a$	$+$	$b/N$	$+$	$c/N^2$	$+$	$d/N^3$
			$a$		$b$		$c$		$d$
3	103.537190082645								
4	57.4074000134889								
5	30.083006000112								
6	8.52435062302512		-202.192		2030.17		-5851.51		7537.7
7	-1.73452473741117		248.055		-4723.54		27466.8		-4649
$\infty$		-24							

Table 3.8:  $1/N$  expansion coefficients for  $\hat{I}_6$  when  $\Gamma = 8$ .

$N$	$\hat{I}_8$	$=$	$a$	$+$	$b/N$	$+$	$c/N^2$	$+$	$d/N^3$
			$a$		$b$		$c$		$d$
3	1073.4573622182								
4	-228.646302651363								
5	-1293.26989559543								
6	-1868.87943727765		1402.47		-68534		384230		-544767
7	-1927.54995720038		13256.9		246350		$1.26146 \times 10^6$		$-1.9673 \times 10^6$

Table 3.9:  $1/N$  expansion coefficients for  $\hat{I}_8$  when  $\Gamma = 8$ .

## 4 Diagrammatic expansions

The proof of the sixth moment sum rule [13] is based on an analysis of the Mayer diagrammatic expansion of the direct correlation function, in the flat space in the bulk. Therefore, it is interesting to study this expansion on the sphere.

The direct correlation function  $c$  is defined by the Ornstein-Zernike equation

$$h(\vec{r}_1, \vec{r}_2) = c(\vec{r}_1, \vec{r}_2) + \rho \int h(\vec{r}_1, \vec{r}_3) c(\vec{r}_3, \vec{r}_2) dS_3, \quad (4.1)$$

where  $\vec{r}_i$  points the direction of a point in the sphere from its center. Since the system is homogeneous, both  $h(\vec{r}_1, \vec{r}_2)$  and  $c(\vec{r}_1, \vec{r}_2)$  depend only the angle  $\theta_{12} = (\vec{r}_1, \vec{r}_2)$ . It is useful to introduce the expansion of  $h$  and  $c$  in Legendre polynomials

$$h(\theta) = \sum_{l=0}^{\infty} h_l P_l(\cos \theta) \quad (4.2)$$

$$h_l = \frac{2l+1}{2} \int_{-1}^1 h(x) P_l(x) dx \quad (4.3)$$

and similarly for  $c$ . The Ornstein-Zernike equation reads

$$h_l = c_l + \frac{N}{2l+1} h_l c_l. \quad (4.4)$$

implying

$$h_l = \frac{c_l}{1 - \frac{N}{2l+1} c_l}. \quad (4.5)$$

Notice also that the moments  $\hat{I}_{2n}$  of the total correlation function are directly related to the coefficients  $h_l$ , since  $P_l(\cos \theta)$  can be expressed as a polynomial in  $\sin^2(\theta/2)$  using the formula  $\cos \theta = 1 - 2 \sin^2(\theta/2)$ .

As explained in detail in [13], the direct correlation function has a renormalized diagrammatic expansion which reads

$$c(\vec{r}_1, \vec{r}_2) = -\Gamma v(\vec{r}_1, \vec{r}_2) + c^{(0)}(\vec{r}_1, \vec{r}_2) + \sum_{s=1}^{\infty} c^{(s)}(\vec{r}_1, \vec{r}_2) \quad (4.6)$$

where  $v(\vec{r}_1, \vec{r}_2)$  is the Coulomb potential between to unit charges.  $c^{(0)}$  is the renormalized “watermelon” Meeron graph

$$c^{(0)}(\vec{r}_1, \vec{r}_2) = \frac{1}{2} K^2(\vec{r}_1, \vec{r}_2) \quad (4.7)$$

where  $K$  is defined by the equation

$$K(\vec{r}_1, \vec{r}_2) = -\Gamma v(\vec{r}_1, \vec{r}_2) - \rho \int \Gamma v(\vec{r}_1, \vec{r}_3) K(\vec{r}_3, \vec{r}_1) dS_3. \quad (4.8)$$

The rest of the expansion of  $c$ ,  $\sum_{s=1}^{\infty} c^{(s)}(\vec{r}_1, \vec{r}_2)$  correspond to the remaining diagrams; see [13] for details.

The coefficients  $K_l$  of the expansion of  $K$  in Legendre polynomials can be obtained from (4.8)

$$K_l = \frac{-\Gamma v_l}{1 + \frac{N\Gamma v_l}{2l+1}}, \quad (4.9)$$

where the coefficients of the Coulomb potential are obtained by solving Poisson equation in the sphere

$$v_l = \frac{2l+1}{2l(l+1)}, \quad l > 0. \quad (4.10)$$

Notice that the coefficient for  $l = 0$  is not defined properly, rather as shown in [20] it is a constant that can be chosen arbitrary. Then

$$K_l = \frac{-\Gamma(2l+1)}{2l(l+1) + N\Gamma}, \quad l > 0. \quad (4.11)$$

In [13] it is proved that the zeroth and second moment of each  $c^{(s)}$  vanishes, therefore they do not contribute to small- $k$  expansion of the Fourier transform of  $c$  up to terms of order  $k^4$ . Then the structure of the Ornstein-Zernike equation in Fourier space guaranties that these terms do not contribute to the expansion of the total correlation function  $h$  up to  $k^8$ . This fixes the value of the moments up to the sixth. Unfortunately, on the sphere this argument breaks down. If one can show that the zeroth and second moment of  $c^{(s)}$  also vanish on the sphere, this fixes the values of  $c_0$ , and  $c_1$ , and from eq. (4.4) one can deduce the value of  $h_0$ ,  $h_1$  and therefore  $\hat{I}_0$  and  $\hat{I}_2$  but unfortunately one cannot deduce anything exact regarding  $h_2$ ,  $h_3$ , and  $\hat{I}_4$ ,  $\hat{I}_6$ .

## 4.1 Second moment

Suppose, as in the plane, that  $c^{(s)}$  has vanishing zeroth and second moment, i.e.  $c_0^{(s)} = 0$ , and  $c_1^{(s)} = 0$ . Then to compute  $c_l$ , eq. (4.6) with the sum ignored implies

$$c_l = -\Gamma v_l + c_l^{(0)} = -\Gamma v_l + \frac{1}{2}m_l. \quad (4.12)$$

The coefficients  $c_l^{(0)} = m_l/2$  are obtained from those of the expansion of  $K^2$ ,

$$K^2(\theta) = \sum_{l=0}^{\infty} m_l P_l(\cos \theta) \quad (4.13)$$

which can be obtained if the expansion of a product of two Legendre functions is known. If

$$P_l(x)P_{l'}(x) = \sum_{l''=0}^{\infty} p_{l''}^{ll'} P_{l''}(x) \quad (4.14)$$

with

$$p_{l''}^{ll'} = \frac{2l''+1}{2} \int_{-1}^1 P_{l''}(x)P_{l'}(x)P_l(x) dx \quad (4.15)$$



then

$$m_{l''} = \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} K_l K_{l'} p_{l''}^{l'l'} . \quad (4.16)$$

The  $p_{l''}^{l'l'}$  can be computed [11] using the recursion equation

$$(2l+1)xP_l(x) = lP_{l-1}(x) + (l+1)P_{l+1}(x) . \quad (4.17)$$

For instance, for  $l'' = 1$ , using  $x = P_1(x)$  in (4.17) and replacing it in (4.15) gives [11]

$$p_1^{l'l'} = \frac{3}{2} \left[ \frac{2(l+1)}{(2l+1)(2l+3)} \delta_{l',l+1} + \frac{2l}{(2l-1)(2l+1)} \delta_{l',l-1} \right] . \quad (4.18)$$

Therefore,

$$\begin{aligned} m_1 &= 3 \sum_{l=0}^{\infty} \frac{2(l+1)}{(2l+1)(2l+3)} K_{l+1} K_l \\ &= 6\Gamma^2 \sum_{l=0}^{\infty} \frac{l+1}{(2(l+1)l + N\Gamma)(2(l+1)(l+2) + N\Gamma)} \\ &= \frac{3\Gamma^2}{2} \sum_{l=0}^{\infty} \left[ \frac{1}{2l(l+1) + N\Gamma} - \frac{1}{2(l+1)(l+2) + N\Gamma} \right] \\ &= \frac{3\Gamma}{2N} . \end{aligned} \quad (4.19)$$

Then, replacing (4.19), into (4.12) and (4.5)

$$h_1 = \frac{-\frac{3\Gamma}{4} \left(1 - \frac{1}{N}\right)}{1 + \frac{N\Gamma}{4} - \frac{\Gamma}{4}} . \quad (4.20)$$

But

$$\begin{aligned} h_1 &= \frac{3}{2} \int h(\theta) (1 - 2 \sin^2(\theta/2)) d(\cos \theta) \\ &= \frac{3}{N} \left( \hat{I}_0 - \frac{4}{N\Gamma} \hat{I}_2 \right) . \end{aligned} \quad (4.21)$$

Since  $\hat{I}_0 = -1$ , we verify that (4.20) is compatible with the sum rule from section 3,  $\hat{I}_2 = N\Gamma/(\Gamma - N\Gamma - 4)$ .

Since the second moment sum rule was proven on firm grounds in section 3, going backwards, this proves indirectly that  $\sum_{s=1}^{\infty} c_1^{(s)} = 0$ .

## 4.2 Higher moments

Although there is no reason to suppose that the contributions from  $\sum_{s=1}^{\infty} c_l^{(s)}$  for  $l = 2$  and  $l = 3$  will vanish for a finite sphere, it is interesting to compute the contribution from  $-\Gamma v_l + c^{(0)}$  to the 4th and 6th moments. Let us define an approximate direct correlation function  $c^{\text{approx}}$  by

$$c^{\text{approx}}(\theta) = \sum_{l=0}^{\infty} c_l^{\text{approx}} P_l(\cos \theta) \quad (4.22)$$

with

$$c_l^{\text{approx}} = -\Gamma v_l + \frac{1}{2} m_l, \quad (4.23)$$

and the corresponding total correlation function  $h^{\text{approx}}$  obtained from Ornstein-Zernike equation

$$h_l^{\text{approx}} = \frac{\frac{-(2l+1)\Gamma}{2l(l+1)} + \frac{m_l}{2}}{1 + \frac{N\Gamma}{2l(l+1)} - \frac{Nm_l}{2(2l+1)}}. \quad (4.24)$$

## 4.3 Fourth moment

The fourth moment is related to the coefficient  $h_2$  by

$$\begin{aligned} h_2 &= \frac{5\rho_b}{N} \int h(\theta) \left[ 1 - \frac{3}{2}(2\sin(\theta/2))^2 + \frac{3}{8}(2\sin(\theta/2))^4 \right] dS \\ &= \frac{5}{N} \left[ \hat{I}_0 - \frac{12}{N\Gamma} \hat{I}_2 + \frac{24}{(N\Gamma)^2} \hat{I}_4 \right], \end{aligned} \quad (4.25)$$

and so

$$\hat{I}_4 = \frac{(N\Gamma)^2}{24} \left[ \frac{Nh_2}{5} + 1 + \frac{12}{\Gamma - N\Gamma - 4} \right]. \quad (4.26)$$

The quantity  $h_2^{\text{approx}}$  is given by

$$h_2^{\text{approx}} = \frac{-\frac{5\Gamma}{12} + \frac{m_2}{2}}{1 + \frac{N\Gamma}{12} - \frac{Nm_2}{10}} \quad (4.27)$$

with

$$m_2 = \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} K_l K_{l'} p_2^{ll'} \quad (4.28)$$

and [11]

$$p_2^{ll'} = \frac{5}{4} \left[ \frac{6(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)} \delta_{l',l+2} + \frac{6(l'+1)(l'+2)}{(2l'+1)(2l'+3)(2l'+5)} \delta_{l,l'+2} - \frac{2}{2l+1} \delta_{ll'} \right]. \quad (4.29)$$

Then

$$\begin{aligned}
m_2 &= \sum_{l=0}^{\infty} \frac{15K_l K_{l+1}(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)} \\
&\quad + \sum_{l=0}^{\infty} \frac{5K_l^2 l(l+1)}{(2l-1)(2l+1)(2l+3)} \\
&= \Gamma^2 f(N\Gamma)
\end{aligned} \tag{4.30}$$

with

$$f(x) = \sum_{l=0}^{\infty} \left[ \frac{15(l+1)(l+2)}{(2l(l+1)+x)(2(l+2)(l+3)+x)(2l+3)} + \frac{5(2l+1)l(l+1)}{(2l(l+1)+x)^2(2l-1)(2l+3)} \right]. \tag{4.31}$$

Replacing this into (4.27) and (4.26) gives an approximation for the fourth moment  $\hat{I}_4^{\text{approx}}$ .

#### 4.4 Sixth moment

Following similar steps, one can obtain an approximation for  $\hat{I}_6$ . The sixth moment is related to  $h_3$  by

$$h_3 = \frac{7}{N} \left[ \hat{I}_0 - \frac{24\hat{I}_2}{N\Gamma} + \frac{120\hat{I}_4}{(N\Gamma)^2} - \frac{160\hat{I}_6}{(N\Gamma)^3} \right] \tag{4.32}$$

Therefore, using (4.26) and the known values of  $\hat{I}_0$  and  $\hat{I}_2$ ,

$$\hat{I}_6 = (N\Gamma)^3 \left[ -\frac{Nh_3}{1120} + \frac{Nh_2}{160} + \frac{1}{40} + \frac{9}{40(\Gamma - N\Gamma - 4)} \right] \tag{4.33}$$

$h_2$  can be approximated by  $h_2^{\text{approx}}$  from last section, and  $h_3$  can be approximated by

$$h_3^{\text{approx}} = \frac{-\frac{7\Gamma}{24} + \frac{m_3}{2}}{1 + \frac{N\Gamma}{24} - \frac{Nm_3}{14}} \tag{4.34}$$

with

$$m_3 = \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} K_l K_{l'} p_3^{ll'} \tag{4.35}$$

and

$$p_3^{ll'} = \frac{7}{4} \int_{-1}^1 (5x^3 - 3x) P_l(x) P_{l'}(x) dx. \tag{4.36}$$

$p_3^{ll'}$  can be computed by applying the recursion equation (4.17) several times to obtain

$$\begin{aligned}
x^3 P_l(x) &= \frac{1}{2l+1} \left[ \frac{l(l-1)(l-2)}{(2l-1)(2l-3)} P_{l-3}(x) + \frac{3l(l^2-2)}{(2l-3)(2l+3)} P_{l-1}(x) \right. \\
&\quad \left. + \frac{(l+1)(l+2)(l+3)}{(2l+3)(2l+5)} P_{l+3}(x) + \frac{3(l+1)((l+1)^2-2)}{(2l-1)(2l+5)} P_{l+1}(x) \right].
\end{aligned} \tag{4.37}$$

Then replacing into (4.36)

$$\begin{aligned}
p_3^{ll'} &= \frac{7}{2(2l+1)} \left[ \frac{5l(l-1)(l-2)}{(2l-1)(2l-3)(2l-5)} \delta_{l',l-3} + \frac{3l(l^2-1)}{(2l-3)(2l+3)(2l-1)} \delta_{l',l-1} \right. \\
&\quad \left. + \frac{5(l+1)(l+2)(l+3)}{(2l+3)(2l+5)(2l+7)} \delta_{l',l+3} + \frac{3(l+1)((l+1)^2-1)}{(2l-1)(2l+5)(2l+3)} \delta_{l',l+1} \right].
\end{aligned} \tag{4.38}$$

Replacing this result into (4.35) one obtains

$$\begin{aligned}
m_3 &= 7\Gamma^2 \sum_{l=0}^{\infty} \frac{(l+1)(l+2)}{(2l+5)(2l(l+1)+N\Gamma)} \left[ \frac{3l}{(2(l+1)(l+2)+N\Gamma)(2l-1)} \right. \\
&\quad \left. + \frac{5(l+3)}{(2(l+3)(l+4)+N\Gamma)(2l+3)} \right] \\
&= \frac{7(2+3N\Gamma)\Gamma}{2N(12+3N\Gamma)}.
\end{aligned} \tag{4.39}$$

Then

$$h_3^{\text{approx}} = \frac{-7(-4+N(4-6\Gamma)+N^2\Gamma)\Gamma}{N(96+4(7N-1)\Gamma+(6-N)N\Gamma^2)}. \tag{4.40}$$

$h_3^{\text{approx}}$  and  $h_2^{\text{approx}}$  can be replaced into (4.33) to obtain an approximation for the sixth moment  $\hat{I}_6^{\text{approx}}$ .

## 4.5 Comparison with numerical results

The expressions for the fourth and sixth moments can be compared to their exact values which have been computed numerically in section 3.2. Tables 4.1, 4.2, 4.3, 4.4 show the exact value of the moments and the approximate value obtained from the diagrammatic technique. For the fourth moment  $\hat{I}_4^{\text{approx}}$  gives an acceptable approximation, on the other hand  $\hat{I}_6^{\text{approx}}$  has accumulated too many errors from  $h_2^{\text{approx}}$  and  $h_3^{\text{approx}}$  and gives a poor agreement with the exact value. However, as  $N \rightarrow \infty$  both  $\hat{I}_4^{\text{approx}}$  and  $\hat{I}_6^{\text{approx}}$  approach the known thermodynamic limit, as expected.

$N$	$\hat{I}_4$	$\hat{I}_4^{\text{approx}}$	$\hat{I}_4$ error	$\hat{I}_6$	$\hat{I}_6^{\text{approx}}$	$\hat{I}_6$ error
2	-1.06666666666667	-0.686993	35.6%	0	0.241677	
3	-0.73469387755102	-0.534741	27.2%	3.30612244897959	1.54027	53.4%
4	-0.552915766738661	-0.42966	22.3%	5.1605471562275	2.52407	51.1%
5	-0.437781621713968	-0.356903	18.5%	6.11292630320115	3.21788	47.4%
6	-0.361584090880502	-0.304426	15.8%	6.56602660174796	3.71182	43.5%
7	-0.307439760694233	-0.26506	13.8%	6.78048920359037	4.07309	39.9%
8	-0.267158562552772	-0.234542	12.2%	6.87498575554362	4.34502	36.8%
9	-0.236094785664912	-0.210236	11.0%	6.9075123159669	4.5552	34.1%
10	-0.211435346122364	-0.190444	9.9%	6.90772192712509	4.72147	31.6%
11	-0.191399568920847	-0.174026	9.1%	6.89103601184945	4.85568	29.5%
12	-0.17480728582518	-0.160195	8.4%	6.86574608931141	4.96591	27.7%
13	-0.160846001528985	-0.148387	7.7%	6.83643498600859	5.05783	26.0%
14	-0.148938895568825	-0.138191	7.2%	6.80567000027031	5.1355	24.5%
$\infty$	0			6		

Table 4.1: Exact value of the moments and their approximation from the diagrammatic technique for  $\Gamma = 4$ .

$N$	$\hat{I}_4$	$\hat{I}_4^{\text{approx}}$	$\hat{I}_4$ error	$\hat{I}_6$	$\hat{I}_6^{\text{approx}}$	$\hat{I}_6$ error
3	1.5	2.12597868844099	41.7%	32.4	7.00428372252487	78.4%
4	1.77112299465241	2.11737368908595	19.5%	29.2857260386672	5.04095606861884	82.8%
5	1.92727455514225	2.10358894768327	9.1%	24.3553395496129	3.85885436737017	84.2%
6	1.97167595506035	2.09127008496943	6.1%	19.3570221116088	3.09958641182161	84.0%
7	1.99917180579183	2.08108076102509	4.1%	15.8157978386411	2.57924397229046	83.7%
8	2.01319334170647	2.07273170306951	3.0%	13.2137920826172	2.20346441160235	83.3%
9	2.02020128573558	2.06584181515105	2.3%	11.2455089383337	1.92065635218756	82.9%
10	2.02414460391682	2.06009120107809	1.8%	9.75173060605997	1.70073355331074	82.6%
11	2.02625080279343	2.0552339688013	1.4%	8.58562698480205	1.52513914797188	82.2%
12	2.02719976576841	2.05108476391051	1.2%	7.65247243254821	1.38187395446424	81.9%
$\infty$	2			0		

Table 4.2: Exact value of the moments and their approximation from the diagrammatic technique for  $\Gamma = 6$ .

$N$	$\hat{I}_4$	$\hat{I}_4^{\text{approx}}$	$\hat{I}_4$ error	$\hat{I}_6$	$\hat{I}_6^{\text{approx}}$	$\hat{I}_6$ error
3	5.9504132231405	7.336835008	23.3%	103.537190082645	-127.1369698	222.8%
4	5.35216744227873	6.1509483	14.9%	57.4074000134889	-71.80581066	225.1%
5	5.31949584221798	5.585392260	5.0%	30.083006000112	-54.66045440	281.7%
6	5.03608123946663	5.254867561	4.3%	8.52435062302512	-46.46113141	645.0%
7	4.90673278031246	5.038212233	2.7%	-1.73452473741117	-41.68739389	2303.4%
8		4.885278337			-38.57330645	
9		4.771577915			-36.38521278	
10		4.683739174			-34.76512324	
11		4.613844211			-33.51802809	
12		4.556906725			-32.52880274	
13		4.509630618			-31.72517539	
14		4.469749993			-31.05951906	
15		4.435655939			-30.49918829	
16		4.406174568			-30.02106637	
17		4.380429327			-29.60832983	
18		4.357752484			-29.24844597	
19		4.337626449			-28.93188851	
20		4.319643825			-28.65128805	
$\infty$	4			-24		

Table 4.3: Exact value of the moments and their approximation from the diagrammatic technique for  $\Gamma = 8$ .

$N$	$\hat{I}_4$	$\hat{I}_4^{\text{approx}}$	$\hat{I}_4$ error	$\hat{I}_6$	$\hat{I}_6^{\text{approx}}$	$\hat{I}_6$ error
	2			3		
2	-0.6666666666666667	-0.6317574181	5.236%	-0.8	-0.752415111	5.95%
3	-0.9	-0.8700339821	3.330%	-1.35	-1.337880192	0.90%
4	-1.0666666666666667	-1.042131679	2.300%	-1.82857142857143	-1.864218645	1.95%
5	-1.19047619047619	-1.170437198	1.683%	-2.23214285714286	-2.312761744	3.61%
6	-1.28571428571429	-1.26919496	1.285%	-2.57142857142857	-2.689601017	4.60%
7	-1.3611111111111111	-1.347327155	1.013%	-2.858333333333333	-3.00618709	5.17%
8	-1.4222222222222222	-1.410579495	0.819%	-3.103030303030303	-3.27362061	5.50%
9	-1.4727272727272727	-1.462780494	0.675%	-3.313636363636363	-3.501250962	5.66%
10	-1.5151515151515152	-1.50656529	0.567%	-3.4965034965035	-3.69658924	5.72%
11	-1.55128205128205	-1.543801073	0.482%	-3.65659340659341	-3.865574853	5.72%
12	-1.58241758241758	-1.57584505	0.415%	-3.7978021978022	-4.012891553	5.66%
13	-1.60952380952381	-1.603706117	0.361%	-3.92321428571429	-4.142245077	5.58%
14	-1.6333333333333333	-1.628149088	0.317%	-4.03529411764706	-4.256585403	5.48%
15	-1.65441176470588	-1.649763932	0.281%	-4.13602941176471	-4.358278405	5.37%
16	-1.67320261437909	-1.669012791	0.250%	-4.22703818369453	-4.449236580	5.26%
17	-1.69005847953216	-1.686262682	0.225%	-4.30964912280702	-4.531018304	5.14%
18	-1.70526315789474	-1.701808689	0.203%	-4.38496240601504	-4.604903343	5.02%
19	-1.71904761904762	-1.715890707	0.184%	-4.4538961038961	-4.671950573	4.90%
20	-1.73160173160173	-1.72870573	0.167%	-4.51722190852626	-4.733042400	4.78%
21	-1.74308300395257	-1.740417020	0.153%	-4.57559288537549	-4.788919165	4.66%
22	-1.7536231884058	-1.751160988	0.140%	-4.6295652173913	-4.840206004	4.55%
23	-1.7633333333333333	-1.761052508	0.129%	-4.67961538461538	-4.88743397	4.44%
24	-1.77230769230769	-1.770188989	0.120%	-4.72615384615385	-4.931056780	4.34%
25	-1.78062678062678	-1.778653568	0.111%	-4.76953601953602	-4.971464120	4.23%
26	-1.78835978835979	-1.786517625	0.103%	-4.81007115489874	-5.008992411	4.14%
27	-1.79556650246305	-1.793842793	0.096%	-4.84802955665025	-5.043933457	4.04%
28	-1.80229885057471	-1.800682557	0.090%	-4.88364849833148	-5.076541501	3.95%
29	-1.80860215053763	-1.807083561	0.084%	-4.91713709677419	-5.107039000	3.86%
30	-1.81451612903226	-1.813086665	0.079%	-4.94868035190616	-5.135621381	3.78%
31	-1.82007575757576	-1.818727815	0.074%	-4.97844251336898	-5.16246097	3.70%
32	-1.825311942959	-1.824038756	0.070%	-5.00656990068755	-5.187710286	3.62%
$\infty$	-2			-6		

Table 4.4: Exact value of the moments and their approximation from the diagrammatic technique for  $\Gamma = 2$ .

## 5 Fluctuation of linear statistics

It was remarked in the Introduction that the sum rules for the moments of  $\rho_{(2)}^T(\vec{r}, \vec{0})$  are related to the average value of the linear statistic  $\sum_{l=1}^N |\vec{r}_l|^{2n}$ . Here we point out that in the case of the soft disk geometry and  $n = 1$  the exact distribution function for this linear statistic is easily evaluated. Two features of the exact distribution are singled out for further study. One relates to the  $O(1)$  correction term to the mean. We are able to deduce the value of this correction to the mean for a general smooth radial linear statistic. The other is that for large  $N$  the exact distribution tends to a Gaussian, with mean and variance as can be anticipated from a Coulomb gas viewpoint [8]. We will show that a mechanism for such Gaussian fluctuations in the case of general rotationally invariant linear statistics can be identified from the viewpoint of the expansions (2.3) and (2.8).

First, with  $\langle \cdot \rangle_{\text{plane}}$  denoting an average with respect to the Boltzmann factor (1.5), we observe that a simple scaling of the coordinates  $\vec{r}_j$  gives

$$\left\langle e^{ik \sum_{j=1}^N |\vec{r}_j|^2} \right\rangle_{\text{plane}} = \left( 1 - \frac{2ik}{\Gamma \pi \rho_b} \right)^{-N - \Gamma N(N-1)/4}. \quad (5.1)$$

Coulomb gas theory [8] tells us that setting  $R = 1$ , so  $\rho_b = N/\pi$ , then taking  $N \rightarrow \infty$ , the distribution function of a radial linear statistic  $A = \sum_{l=1}^N g(r_l)$  should become the Gaussian

$$\left\langle e^{ik \sum_{j=1}^N g(r_l)} \right\rangle_{\text{plane}} \underset{N \rightarrow \infty}{\sim} e^{ik \mathcal{M}_N} e^{-k^2 \sigma^2 / 2} \quad (5.2)$$

with

$$\begin{aligned} \mathcal{M}_N &= 2N \int_0^1 r g(r) dr + O(1) \\ \sigma^2 &= \frac{1}{\Gamma} \int_0^1 r (g'(r))^2 dr + o(1). \end{aligned} \quad (5.3)$$

### 5.1 Universal correction to the mean $\mathcal{M}_N$

According to (5.1), with  $\rho_b = N/\pi$ ,

$$\left\langle e^{ik \sum_{j=1}^N |\vec{r}_j|^2} \right\rangle_{\text{plane}} = \exp \left( \frac{ikN}{2} + \frac{2ik}{\Gamma} \left( 1 - \frac{\Gamma}{4} \right) - \frac{k^2}{2\Gamma} + O\left(\frac{1}{N}\right) \right). \quad (5.4)$$

This is of the form (5.2) with

$$\mathcal{M}_N = \frac{N}{2} + \frac{2}{\Gamma} \left( 1 - \frac{\Gamma}{4} \right) \quad (5.5)$$

$$\sigma^2 = \frac{1}{\Gamma}, \quad (5.6)$$

and (5.5), (5.6) are in turn consistent with (5.3) in the case  $g(r) = r^2$ . The expression (5.5) is an identity for all  $N$ . But with  $g(r) = r^2$ ,

$$\mathcal{M}_N = \int_{\mathbb{R}^2} r^2 \rho_{(1)}(r) d^2 \vec{r}. \quad (5.7)$$



Writing  $\tilde{\rho}_b = \rho_b \chi_{0 < r < R}$ , where  $\chi_A = 1$  if  $A$  is true, and  $\chi_A = 0$  otherwise, this can be written

$$\mathcal{M}_N = \frac{\pi}{2} R^4 \rho_b + \int_{\mathbb{R}^2} r^2 (\rho_{(1)}(\vec{r}) - \tilde{\rho}_b) d^2 \vec{r}. \quad (5.8)$$

Recalling that in (5.5),  $\rho_b = N/\pi$  and  $R = 1$ , comparison of (5.8) and (5.5) gives the sum rule

$$\int_{\mathbb{R}^2} r^2 (\rho_{(1)}(\vec{r}) - \tilde{\rho}_b) d^2 \vec{r} = R^2 \frac{2}{\Gamma} \left(1 - \frac{\Gamma}{4}\right). \quad (5.9)$$

We remark that an alternative derivation of (5.9), valid for  $\Gamma$  even, is to use (2.4) and (2.9), together with the recurrence relation for the gamma function. We remark too that (5.9) is the soft disk version of the sum rule for the 2dOCP in a disk with hard wall boundary conditions [7],

$$\rho_{(1)}(0) - \left(1 - \frac{\Gamma}{4}\right) \rho_b = -\frac{\Gamma \rho_b^2 \pi^2}{N} \int_0^R r^3 (\rho_{(1)}(R-r) - \rho_b) dr, \quad (5.10)$$

where  $\rho_{(1)}(r)$  is measured inward from the boundary. This latter viewpoint allows the  $N \rightarrow \infty$  limit of (5.9) to be formulated. Thus on the LHS change to polar coordinates, and further change variables  $r \mapsto R - r$ . Defining  $\rho_{(1)}^{\text{sw}}(r) = \rho_{(1)}(R - r)$ , corresponding to the density as measured inward from  $r = R$  in the soft wall disk OCP, and noting that charge neutrality requires

$$\int_{-\infty}^R (R - r) (\rho_{(1)}^{\text{sw}}(r) - \tilde{\rho}_b) dr = 0 \quad (5.11)$$

we see that

$$\begin{aligned} \int_0^\infty r^3 (\rho_{(1)}(r) - \tilde{\rho}_b) dr &= \int_{-\infty}^R (R - r)^3 (\rho_{(1)}^{\text{sw}}(r) - \tilde{\rho}_b) dr \\ &= \int_{-\infty}^R (-2R^2 r + 3Rr^2 - r^3) (\rho_{(1)}^{\text{sw}}(r) - \tilde{\rho}_b) dr \\ &= -2R^2 \int_{-\infty}^\infty r (\rho_{(1)}^{\text{sw}}(r) - \tilde{\rho}_b) dr + O(R) \end{aligned} \quad (5.12)$$

(compare with working below (3.17) of [19]). Comparison with (5.9) we thus have that in the thermodynamic limit

$$-2\pi\Gamma \int_{-\infty}^\infty r (\rho_{(1)}^{\text{sw}}(r) - \tilde{\rho}_b) dr = \left(1 - \frac{\Gamma}{4}\right). \quad (5.13)$$

The sum rule (5.13) should be compared against the contact theorem [7] for the hard wall plasma,

$$\rho_{(1)}^{\text{hw}}(0) - 2\pi\Gamma \int_0^\infty r (\rho_{(1)}^{\text{hw}}(r) - \tilde{\rho}_b) dr = \left(1 - \frac{\Gamma}{4}\right), \quad (5.14)$$

where  $\rho_{(1)}^{\text{hw}}(r)$  denotes the density as measured from the boundary.

The above working suggests that the first two terms in the asymptotic expansion of  $\mathcal{M}_N$  in the case of  $g(r) = r^m$ , for  $m \in \mathbb{Z}^+$ , admit a form analogous to that exhibited in (5.5). First, with this choice of  $g(r)$ , analogous to (5.8) we have

$$\mathcal{M}_N = \frac{2\pi}{m+2} R^{m+2} \rho_b + 2\pi \int_0^\infty r^{m+1} (\rho_{(1)}(r) - \tilde{\rho}_b) dr.$$

Manipulation of the integral as in (5.12) now shows

$$\mathcal{M}_N = \frac{2\pi}{m+2} R^{m+2} \rho_b - 2\pi p R^m \int_{-\infty}^{\infty} r(\rho_{(1)}^{\text{sw}}(r) - \tilde{\rho}_b) dr + O(R^{m-1} \rho_b^{-1/2}).$$

Substituting of (5.13) we then obtain

$$\mathcal{M}_N = \frac{2\pi}{m+2} R^{m+2} \rho_b + \frac{m}{\Gamma} \left(1 - \frac{\Gamma}{4}\right) R^m + O(R^{m-1} \rho_b^{-1/2}). \quad (5.15)$$

In particular, with  $R = 1$  and  $\rho_b = N/\pi$ , we have

$$\mathcal{M}_N = \frac{2N}{m+2} + \frac{m}{\Gamma} \left(1 - \frac{\Gamma}{4}\right) + O\left(\frac{1}{N^{1/2}}\right). \quad (5.16)$$

In this setting write  $\rho_{(1)}(r) = (N/\pi)\chi_{0 < r < 1} + \kappa(r) + O(N^{-1/2})$ . Then it follows by comparing the definition of  $\mathcal{M}_N$  in the case  $g(r) = r^m$  to (5.16) that

$$2\pi \int_0^{\infty} r^{m+1} \kappa(r) dr = \frac{m}{\Gamma} \left(1 - \frac{\Gamma}{4}\right).$$

And it follows from this that  $\kappa(r)$  is the distribution supported at  $r = 1$  given by

$$\kappa(r) = \frac{1}{2\pi\Gamma} \left(1 - \frac{\Gamma}{4}\right) \frac{1}{r} \delta'(r-1). \quad (5.17)$$

Thus for  $g(r)$  smooth in the neighbourhood of  $r = 1$ , and with  $R = 1$  and  $\rho_b = N/\pi$ , we have that for large  $N$

$$\mathcal{M}_N = 2\pi N \int_0^1 r g(r) dr + \frac{1}{\Gamma} \left(1 - \frac{\Gamma}{4}\right) g'(1) + O(N^{-1/2}). \quad (5.18)$$

## 5.2 Derivation using (2.9)

We will now consider the structure of the Gaussian fluctuation formula (5.2) itself. It can readily be deduced at  $\Gamma = 2$  as a consequence of the expansion formula (2.9) for  $I_{N,\Gamma}[g]$  [8]. Thus then  $p = 1$  and  $c_\mu^{(N)}(1) = 1$  for  $\mu = \delta_N$ , and  $c_\mu^{(N)}(1) = 0$  otherwise. It is interesting to probe how (2.9) can lead to (5.2) for general  $\Gamma = 4p + 2$ . We then have

$$\left\langle e^{ik \sum_{j=1}^N a(r_j)} \right\rangle_{\text{plane}} = \frac{I_{N,\Gamma}[e^{ika(r)}]}{I_{N,\Gamma}[1]} \quad (5.19)$$

where, with  $G_{\mu_l}$  specified by (2.5)

$$I_{N,\Gamma}[e^{ika(r)}] = N! \pi^N \sum_{\mu} \left( c_\mu^{(N)}(2p+1) \right)^2 \prod_{l=1}^N G_{\mu_l} [e^{-N\Gamma r^2/2} e^{ika(r^2)}], \quad (5.20)$$

and according to (2.6)

$$\frac{\Gamma}{2}(N-1) \geq \mu_1 > \mu_2 > \cdots > \mu_N \geq 0 \quad (5.21)$$

with  $|\mu| = \Gamma N(N-1)/4$ . Let's introduce the scaled variables  $\tilde{\mu}_l := 2\mu_l/N\Gamma$  so that to leading order in  $N$

$$1 \geq \tilde{\mu}_1 > \tilde{\mu}_2 > \cdots > \tilde{\mu}_N \geq 0.$$

In terms of these scaled variables, and upon the change of variables  $r^2 = s$ , we see

$$G_{\mu_l}[e^{-N\Gamma r^2/2} e^{ika(r^2)}] = \int_0^\infty e^{-(\Gamma N/2)(s - \tilde{\mu}_l \log s) + ika(\sqrt{s})} ds.$$

For large  $N$  the maximum of the exponent occurs at  $s = \tilde{\mu}_l$ . Expanding the integrand about this point and completing the square shows that for large  $N$

$$G_{\mu_l}[e^{-N\Gamma r^2/2} e^{ika(r^2)}] \sim G_{\mu_l}[e^{-N\Gamma r^2/2}] \exp \left( ika(\sqrt{\tilde{\mu}_l}) - \frac{k^2}{4\Gamma N} (a'(\sqrt{\tilde{\mu}_l}))^2 \right). \quad (5.22)$$

To proceed further we hypothesize that the values of  $\{\tilde{\mu}_l\}$  for which the  $c_\mu^{(N)}(2p+1)$  in (2.9) are nonzero are uniformly distributed (or at least this situation dominates). Then we have

$$\begin{aligned} & \prod_{l=1}^N G_{\mu_l}[e^{-N\Gamma r^2/2} e^{ika(r^2)}] \\ & \sim \prod_{l=1}^N G_{\mu_l}[e^{-N\Gamma r^2/2}] \exp \left( ikN \int_0^1 a(\sqrt{u}) du - \frac{k^2}{4\Gamma} \int_0^1 (a'(\sqrt{u}))^2 du \right). \end{aligned} \quad (5.23)$$

Substituting in (5.20) then substituting the result in (5.19) reclaims (5.2).

### 5.3 Test near $\Gamma = 2$

In this subsection we verify (5.16) for  $\Gamma = 2$  and for  $\Gamma$  close to 2, where explicit analytical calculations of  $\mathcal{M}_N$  can be done. When  $\Gamma = 2$ , the density profile is [9, §15]

$$\rho_{(1)}(r) = \rho_b e^{-\pi \rho_b r^2} \sum_{k=0}^{N-1} \frac{(\pi \rho_b r^2)^k}{k!} = \frac{N\Gamma(N, Nr^2)}{\pi(N-1)!}, \quad (5.24)$$

where  $\pi \rho_b = N$ . Then, with  $m = 2n$ ,

$$\mathcal{M}_N = \int_{\mathbb{R}^2} r^{2n} \rho_{(1)}(r) d^2 \vec{r} = \frac{1}{N^n} \sum_{k=0}^{N-1} \frac{(k+n)!}{k!} = \frac{N(N+n)!}{N^n(1+n)N!}. \quad (5.25)$$

Now, when  $N \rightarrow \infty$ ,

$$\mathcal{M}_N|_{\Gamma=2} = \frac{N}{n+1} + \frac{n}{2} + \frac{n(n-1)(3n+2)}{N} + O(1/N^2) \quad (5.26)$$

which satisfies (5.16) when  $\Gamma = 2$ . It is interesting to notice that there are no  $O(N^{-1/2})$  corrections to  $\mathcal{M}_N$  at  $\Gamma = 2$ . This is, however, a special feature of the  $\Gamma = 2$  case, as for general  $\Gamma$  there will be non zero  $O(N^{-1/2})$  corrections (see sec. 5.4). A similar situation happens for the finite-size corrections to the free energy of the 2dOCP in the soft disk at  $\Gamma = 2$  [19].

For  $\Gamma$  close to 2, one can perform an expansion in powers of  $\Gamma - 2$  of the density profile, to compute  $\mathcal{M}_N$ , similar to a computation done in [12]. To the linear order in  $\Gamma - 2$ , the density is given by  $\rho_{(1)}(\vec{r}; \Gamma) = \rho_{(1)}(\vec{r}; 2) - \frac{\Gamma-2}{2} \langle \hat{\rho}(\vec{r}) \beta H \rangle^T + \mathcal{O}((\Gamma - 2)^2)$ , where the truncated average is taken with the Boltzmann factor (1.5) at  $\Gamma = 2$ ,  $H$  is the potential energy of the 2dOCP and  $\hat{\rho}(\vec{r})$  is the microscopic density. Explicitly,

$$\begin{aligned} \langle \hat{\rho}(\vec{r}) \beta H \rangle^T &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} [\rho_{(3)}(\vec{r}_1, \vec{r}_2, \vec{r}) - \rho_{(1)}(\vec{r}) \rho_{(2)}(\vec{r}_1, \vec{r}_2)] v(\vec{r}_1, \vec{r}_2) d^2 \vec{r}_1 d^2 \vec{r}_2 + \pi \rho_b r^2 \rho_{(1)}(\vec{r}) \\ &+ 2 \int_{\mathbb{R}^2} \rho_{(2)}(\vec{r}_1, \vec{r}) v(\vec{r}_1, \vec{r}) d^2 \vec{r}_1 + \pi \rho_b \int_{\mathbb{R}^2} r_1^2 [\rho_{(2)}(\vec{r}_1, \vec{r}) - \rho_{(1)}(\vec{r}_1) \rho_{(1)}(\vec{r})] d^2 \vec{r}_1 \end{aligned} \quad (5.27)$$

where the density and correlations in the RHS are evaluated at  $\Gamma = 2$  and  $v(\vec{r}, \vec{r}')$  is the Coulomb pair potential (1.1). At  $\Gamma = 2$ , the correlation functions have the simple structure [12] [9, §15.3]

$$\rho_{(\ell)}(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_\ell) = \rho_b^\ell \det(K(z_i, z_j))_{1 \leq i, j \leq \ell}, \quad (5.28)$$

with

$$K(z, z') = e^{-(|z|^2 + |z'|^2)/2} \sum_{k=0}^{N-1} \frac{(z \bar{z}')^k}{k!}, \quad (5.29)$$

and  $z = \sqrt{\pi \rho_b} r e^{i\phi}$ , where  $(r, \phi)$  are the polar coordinates of  $\vec{r}$ . To perform the integrals in (5.27), it is convenient to expand the Coulomb potential (1.1) in a Fourier series in the polar angle

$$v(\vec{r}_1, \vec{r}_2) = \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{r_{<}}{r_{>}} \right)^m \cos[m(\phi_1 - \phi_2)] - \log \frac{r_{>}}{L}, \quad (5.30)$$

with  $r_{>} = \max(r_1, r_2)$  and  $r_{<} = \min(r_1, r_2)$ . Replacing (5.28) and (5.30) in (5.27), after much algebra, one obtains

$$\begin{aligned} \rho_{(1)}(\vec{r}; \Gamma) &= \rho_{(1)}(\vec{r}; 2) \\ &- (\Gamma - 2) \rho_b e^{-|z|^2} \left\{ \sum_{k_1=0}^{N-1} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{N-1} \frac{|z|^{2k_2}}{2k_1! (k_2!)^2} \mathcal{J}(k_1, k_2) + \sum_{k_1=0}^{N-1} \sum_{k_2=k_1+1}^{N-1} \frac{\mathcal{I}(k_1, k_2)}{k_1! k_2! (k_2 - k_1)} \left[ \frac{|z|^{2k_2}}{k_2!} + \frac{|z|^{2k_1}}{k_1!} \right] \right. \\ &- \sum_{k_1=0}^{N-1} \sum_{k_2=k_1+1}^{N-1} \frac{|z|^{2k_1} \gamma(k_2 + 1, |z|^2) + |z|^{2k_2} \Gamma(k_1 + 1, |z|^2)}{(k_2 - k_1) k_1! k_2!} \\ &- \sum_{k_2=0}^{N-1} \frac{|z|^{2k_2}}{2 k_2!} \sum_{k_1=0}^{N-1} \frac{\gamma(k_1, |z|^2) \log(|z|^2) + \int_{|z|^2}^{\infty} e^{-t} t^{k_1} \log t dt}{k_1!} - \sum_{k_1=0}^{N-1} \frac{|z|^{2k_1}}{2 k_1!} [k_1 + 1 - |z|^2] \\ &\left. + \sum_{k_1=0}^{N-1} \frac{|z|^{2k_1}}{2 (k_1!)^2} \left[ \gamma(k_1 + 1, |z|^2) \log(|z|^2) + \int_{|z|^2}^{\infty} e^{-t} t^{k_1} \log t dt \right] \right\} + \mathcal{O}((\Gamma - 2)^2), \end{aligned} \quad (5.31)$$

where we have defined

$$\mathcal{I}(k_1, k_2) = \iint_{0 \leq t_2 < t_1} e^{-t_1 - t_2} t_1^{k_1} t_2^{k_2} dt_1 dt_2, \quad (5.32)$$

and

$$\mathcal{J}(k_1, k_2) = \int_0^\infty \int_0^\infty e^{-t_1-t_2} t_1^{k_1} t_2^{k_2} \log(\max(t_1, t_2)) dt_1 dt_2. \quad (5.33)$$

Some useful properties, as well as the asymptotic expansion when  $k_1 \rightarrow \infty$  and  $k_2 \rightarrow \infty$  of  $\mathcal{I}(k_1, k_2)$  and  $\mathcal{J}(k_1, k_2)$ , are discussed in the appendix.

Consequently, with  $\pi\rho_b = N$ , the  $2n$ -moment of the density profile is

$$\begin{aligned} \mathcal{M}_N &= \mathcal{M}_N|_{\Gamma=2} - (\Gamma - 2)N^{-n} \left\{ \frac{n}{2} \sum_{k_1=0}^{N-1} \frac{(k_1 + n)!}{k_1!} \right. \\ &+ \sum_{k_1=0}^{N-1} \sum_{k_2=k_1+1}^{N-1} \frac{\frac{(k_1+n)!}{k_1!} \mathcal{I}(k_1, k_2) - \mathcal{I}(k_1 + n, k_2) + \frac{(k_2+n)!}{k_2!} \mathcal{I}(k_1, k_2) - \mathcal{I}(k_1, k_2 + n)}{k_1! k_2! (k_2 - k_1)} \Big\} \\ &+ \sum_{k_1=0}^{N-1} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{N-1} \frac{1}{2 k_1! k_2!} \left[ \frac{(k_1 + n)!}{k_1!} \mathcal{J}(k_1, k_2) - \mathcal{J}(k_1 + n, k_2) \right] + O((\Gamma - 2)^2). \end{aligned} \quad (5.34)$$

In the last term of (5.34) it is convenient to use the recurrence relation (A.6) and write

$$\begin{aligned} \frac{(k_1 + n)!}{k_1!} \mathcal{J}(k_1, k_2) - \mathcal{J}(k_1 + n, k_2) &= - \sum_{\ell=0}^{n-1} \frac{(k_1 + n)!}{k_1! (k_1 + 1 + \ell)!} \mathcal{I}(k_1, k_2) \\ &- \sum_{\ell=0}^{n-1} \frac{(k_1 + n)!}{(k_1 + 1 + \ell)!} \left[ \mathcal{I}(k_1 + \ell, k_2) - \frac{(k_1 + \ell)!}{k_1!} \mathcal{I}(k_1, k_2) \right]. \end{aligned} \quad (5.35)$$

Consequently  $\mathcal{M}_N$  is given by  $\mathcal{M}_N = \mathcal{M}_N|_{\Gamma=2} - (\Gamma - 2)(\tilde{m}_1 + \tilde{m}_2 + \tilde{m}_3) + O((\Gamma - 2)^2)$  with

$$\tilde{m}_1 = \frac{N^{-n}}{2} \left[ \sum_{k_1=0}^{N-1} n \frac{(k_1 + n)!}{k_1!} - \sum_{k_1=0}^{N-1} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{N-1} \frac{1}{k_1! k_2!} \sum_{\ell=0}^{n-1} \frac{(k_1 + n)!}{k_1! (k_1 + 1 + \ell)!} \mathcal{I}(k_1, k_2) \right], \quad (5.36)$$

$$\tilde{m}_2 = -\frac{N^{-n}}{2} \sum_{k_1=0}^{N-1} \sum_{\substack{k_2=0 \\ k_2 \neq k_1}}^{N-1} \frac{1}{k_1! k_2!} \sum_{\ell=0}^{n-1} \frac{(k_1 + n)!}{(k_1 + 1 + \ell)!} \left[ \mathcal{I}(k_1 + \ell, k_2) - \frac{(k_1 + \ell)!}{k_1!} \mathcal{I}(k_1, k_2) \right], \quad (5.37)$$

and

$$\tilde{m}_3 = N^{-n} \sum_{k_1=0}^{N-1} \sum_{k_2=k_1+1}^{N-1} \frac{\frac{(k_1+n)!}{k_1!} \mathcal{I}(k_1, k_2) - \mathcal{I}(k_1 + n, k_2) + \frac{(k_2+n)!}{k_2!} \mathcal{I}(k_1, k_2) - \mathcal{I}(k_1, k_2 + n)}{k_1! k_2! (k_2 - k_1)}. \quad (5.38)$$

In (5.36) it is convenient to split the sum on  $k_1$  and  $k_2$  into two sums for  $k_1 < k_2$  and  $k_1 > k_2$  and in the latter use (A.3) to find  $\tilde{m}_1 = \tilde{m}_1^a + \tilde{m}_1^b$  with

$$\tilde{m}_1^a = \frac{N^{-n}}{2} \sum_{k_1=0}^{N-1} \left( \frac{(k_1 + n)! n}{k_1!} - k_1 \sum_{\ell=0}^{n-1} \frac{(k_1 + n)!}{k_1! (k_1 + 1 + \ell)!} \right), \quad (5.39)$$

and

$$\tilde{m}_1^b = -\frac{N^{-n}}{2} \sum_{k_1=0}^{N-1} \sum_{k_2=k_1+1}^{N-1} \frac{\mathcal{I}(k_1, k_2)}{k_1!k_2!} \sum_{\ell=0}^{n-1} \left( \frac{(k_1+n)!}{k_1!(k_1+1+\ell)} - \frac{(k_2+n)!}{k_2!(k_2+1+\ell)} \right). \quad (5.40)$$

In (5.39), the expression inside the sum is a polynomial of  $k_1$  of order  $n-1$ , therefore to obtain the order  $O(N^0)$  when  $N \rightarrow \infty$  of  $\tilde{m}_1^a$  it is sufficient to consider the leading order of the polynomial and replace the sum over  $k_1$  by an integral,

$$\begin{aligned} \tilde{m}_1^a &= \frac{N^{-n}}{2} \int_0^N dk_1 \sum_{\ell=0}^{n-1} [(\ell+1)k_1^{n-1} + O(k_1^{n-2})] \\ &= \frac{n+1}{4} + O(1/N). \end{aligned} \quad (5.41)$$

In the appendix it is shown that, for large  $k_1$  and  $k_2$ ,

$$\frac{\mathcal{I}(k_1, k_2)}{k_1!k_2!} \sim \frac{1}{2} \operatorname{erfc} \left( \frac{k_2 - k_1}{\sqrt{2(k_1 + k_2)}} \right). \quad (5.42)$$

To obtain the leading order contribution in  $N$  of  $\tilde{m}_1^b$  one should replace the sums over  $k_1$  and  $k_2$  in (5.40) by integrals, and realise from (5.42) that the leading contribution will be obtained for  $k_1$  and  $k_2$  laying in a strip following the line  $k_1 = k_2$ , with  $k_1 < k_2$ , and of width of order  $\sqrt{N}$ . With this in mind, it is useful to do a change of variables in the integrals, from  $k_1$  and  $k_2$  to  $k_+ = k_1 + k_2$ , which goes along the strip, and  $u = (k_2 - k_1)/\sqrt{2k_+}$ , which goes perpendicular to the strip. The integral over  $u$  will give  $O(1)$  contributions, due to the fast decay of the erfc function, while the one over  $k_+$  will give a contribution of order  $O(N^n)$ . To obtain this, it is sufficient to keep the leading order in  $k_+$  in the integrand. In (5.40), the expression in the sum over  $\ell$  is a polynomial of  $k_1$  and  $k_2$  that vanishes when  $k_1 = k_2$ , therefore it can be written as

$$\sum_{\ell=0}^{n-1} \left( \frac{(k_1+n)!}{k_1!(k_1+1+\ell)} - \frac{(k_2+n)!}{k_2!(k_2+1+\ell)} \right) = -(k_2 - k_1)Q(k_1, k_2), \quad (5.43)$$

with  $Q(k_1, k_2)$  a polynomial in  $k_1$  and  $k_2$  of order  $n-2$ . Actually, to the leading order,

$$\begin{aligned} Q(k_1, k_2) &= n \sum_{r=0}^{n-2} k_1^{n-2-r} k_2^r + \text{polynomial in } k_1 \text{ and } k_2 \text{ of order } n-3 \\ &= n(n-1) \left( \frac{k_+}{2} \right)^{n-2} + O(k_+^{n-3}). \end{aligned} \quad (5.44)$$

Then, replacing in (5.40),

$$\begin{aligned} \tilde{m}_1^b &= -\frac{N^{-n}}{2} \sum_{k_1=0}^{N-1} \sum_{k_2=k_1+1}^{N-1} \frac{\mathcal{I}(k_1, k_2)}{k_1!k_2!} (k_2 - k_1)Q(k_1, k_2) \\ &= \frac{N^{-n}}{4} \int_0^\infty du \int_0^{2N} dk_+ u \operatorname{erfc}(u) [n(n-1)2^{-(n-2)}k_+^{n-1} + O(k_+^{n-2})] + O(1/\sqrt{N}) \\ &= \frac{n-1}{4} + O(1/\sqrt{N}), \end{aligned} \quad (5.45)$$

where the integral  $\int_0^\infty u \operatorname{erfc}(u) du = 1/4$  was used. Putting together the last results, we find

$$\tilde{m}_1 = \frac{n}{2} + O(1/\sqrt{N}). \quad (5.46)$$

Regarding the other contributions to  $\mathcal{M}_N$ , it can be noticed using the recurrence relation (A.5) that  $\tilde{m}_2 + \tilde{m}_3 = 0$  when  $n = 0, 1, 2$ , and 3. For  $n > 3$ , the leading order of these terms can be obtained following similar steps as the ones done for  $\tilde{m}_1$ . The sums over  $k_1$  and  $k_2$  are replaced by integrals over the variables  $k_+$  and  $u$  defined above, and only the leading order in  $k_+$  is kept. From (A.13), one can obtain that, for  $k_+$  large,

$$\frac{\mathcal{I}(k_1 + \ell, k_2) - \frac{(k_1 + \ell)!}{k_1!} \mathcal{I}(k_1, k_2)}{k_2! (k_1 + \ell)!} = \frac{\ell e^{-u^2}}{\sqrt{2\pi k_+}} + O(1/k_+). \quad (5.47)$$

Substituting into (5.37) shows

$$\begin{aligned} \tilde{m}_2 &= -N^{-n} 2 \int_{-\infty}^{\infty} du \int_0^{2N} \frac{dk_+ \sqrt{2k_+}}{2} \sum_{l=0}^{n-1} \frac{\ell e^{-u^2}}{\sqrt{2\pi k_+}} \left(\frac{k_+}{2}\right)^{n-1} + O(1/\sqrt{N}) \\ &= -\frac{n-1}{4} + O(1/\sqrt{N}) \end{aligned} \quad (5.48)$$

For  $\tilde{m}_3$ , use

$$\frac{\frac{(k_1+n)!}{k_1!} \mathcal{I}(k_1, k_2) - \mathcal{I}(k_1 + n, k_2)}{k_1! k_2!} = \frac{k_+^n n e^{-u^2}}{2^n \sqrt{2\pi k_+}} \left[ -1 + \frac{(n-1)u}{\sqrt{2k_+}} + O(1/k_+) \right], \quad (5.49)$$

and

$$\frac{\frac{(k_2+n)!}{k_2!} \mathcal{I}(k_1, k_2) - \mathcal{I}(k_1, k_2 + n)}{k_1! k_2!} = \frac{k_+^n n e^{-u^2}}{2^n \sqrt{2\pi k_+}} \left[ 1 + \frac{(n-1)u}{\sqrt{2k_+}} + O(1/k_+) \right], \quad (5.50)$$

to obtain

$$\begin{aligned} \tilde{m}_3 &= N^{-n} \int_0^\infty du \int_0^{2N} \frac{dk_+ \sqrt{2k_+}}{2} \left(\frac{k_+}{2}\right)^n \frac{2e^{-u^2} n(n-1)u}{u \sqrt{2\pi k_+} 2k_+} + O(1/\sqrt{N}) \\ &= \frac{n-1}{4} + O(1/\sqrt{N}). \end{aligned} \quad (5.51)$$

Therefore  $\tilde{m}_2 + \tilde{m}_3 = O(1/\sqrt{N})$  do not contribute to order  $O(1)$  in  $\mathcal{M}_N$ . Summing up all results,

$$\mathcal{M}_N = \frac{N}{n+1} + \frac{n}{2} - (\Gamma-2) \frac{n}{2} + O(1/\sqrt{N}) + O((\Gamma-2)^2). \quad (5.52)$$

This result is in agreement with the general formula (5.16) when it is expanded around  $\Gamma = 2$  to the linear order in  $\Gamma - 2$  with  $m = 2n$ .

## 5.4 Numerical results for $\Gamma = 4, 6$ , and 8

For  $\Gamma$  even, by application of (2.4) or (2.9), the  $2n$ -moment of the density can be expressed as

$$\mathcal{M}_N = \frac{(N\Gamma/2)^{-n}}{Z_{\text{soft}}} \sum_{\mu} \frac{(c_{\mu}^{(N)}(\Gamma/2))^2}{\prod_i m_i!} \prod_{\ell=1}^N \mu_{\ell}! \sum_{k=1}^N \frac{(\mu_k + n)!}{\mu_k!}, \quad (5.53)$$

with

$$Z_{\text{soft}} = \sum_{\mu} \frac{(c_{\mu}^{(N)}(\Gamma/2))^2}{\prod_i m_i!} \prod_{\ell=1}^N \mu_{\ell}!, \quad (5.54)$$

which is the partition function of the 2dOCP in the soft disk, up to a multiplicative constant. Tables 5.1–5.9 show the numerical evaluation of  $\mathcal{M}_N$  as well as a fit to  $\mathcal{M}_N = aN + b + cN^{-1/2} + d/N$ , for  $\Gamma = 4, 6$ , and 8, and  $n = 2, 3$ , and 4. These numerical results, and the proposed fit, show that (5.16) is indeed satisfied. The convergence is very good for all moments for  $\Gamma = 4$  and 6. For  $\Gamma = 8$ , unfortunately we were not able to perform the calculations beyond  $N = 7$  particles, the results presented have not yet converged to the  $N \rightarrow \infty$  expected value. Notice that, contrary to the case  $\Gamma = 2$ , there are non-zero  $O(N^{-1/2})$  finite-size corrections to  $\mathcal{M}_N$ .

$N$	$\mathcal{M}_N$	$=$	$aN$	$+$	$b$	$+$	$c/\sqrt{N}$	$+$	$d/N$
			$a$		$b$		$c$		$d$
2	0.8125								
3	1.126262								
4	1.44563743218807								
5	1.76891109591098		0.336437		-0.0742714		0.458953		-0.221263
6	2.09454890418255		0.333419		-0.00242266		0.269645		-0.0817588
7	2.42171814295119		0.332974		0.0108746		0.230301		-0.0491322
8	2.74996856529295		0.33338		-0.003688474		0.277614		-0.0922608
9	3.07901955876735		0.333458		-0.00698637		0.289203		-0.103695
10	3.40868118671838		0.333396		-0.00397741		0.277889		-0.0917455
11	3.73882025776555		0.333353		-0.00169446		0.268779		-0.0815306
12	4.06934089384864		0.333341		-0.000983289		0.265786		-0.0779911
13	4.4001722794716		0.333341		-0.000982974		0.265784		-0.0779894
14	4.73126081887937		0.333342		-0.00105077		0.266097		-0.0783948
$\infty$			1/3		0				

Table 5.1: Fourth moment ( $n = 2$ ) of the density when  $\Gamma = 4$ .



$N$	$\mathcal{M}_N$	$=$	$aN$	$+$	$b$	$+$	$c/\sqrt{N}$	$+$	$d/N$
			$a$		$b$		$c$		$d$
2	0.890625								
3	1.08333333333333								
4	1.29792043399638								
5	1.52318930281443		0.249835		0.0492187		0.53598		-0.0745192
6	1.75432075208484		0.246695		0.123999		0.338947		0.0706775
7	1.98916255005449		0.248062		0.0831756		0.459736		-0.0294887
8	2.22660014721048		0.249434		0.0339754		0.619579		-0.175195
9	2.46595156197257		0.249738		0.021223		0.664392		-0.219408
10	2.70676308425724		0.249748		0.0207354		0.666226		-0.221345
11	2.94871984780432		0.249774		0.0193389		0.671799		-0.227593
12	3.19159601492947		0.249824		0.0163325		0.684451		-0.242556
13	3.43522469549061		0.249871		0.0132299		0.69815		-0.259556
14	3.67947926593368		0.249905		0.0108374		0.709186		-0.273865
$\infty$			0.25		0				

Table 5.2: Sixth moment ( $n = 3$ ) of the density when  $\Gamma = 4$ .

$N$	$\mathcal{M}_N$	$=$	$aN$	$+$	$b$	$+$	$c/\sqrt{N}$	$+$	$d/N$
			$a$		$b$		$c$		$d$
2	1.21875								
3	1.25420875420875								
4	1.36950440777577								
5	1.51576558282311		0.188419		0.468569		-0.269131		1.12729
6	1.67718058035561		0.189303		0.447536		-0.213712		1.08645
7	1.84742557965232		0.194189		0.301703		0.217776		0.728635
8	2.02343313425351		0.197002		0.20078		0.545661		0.42975
9	2.20346793019457		0.197864		0.164668		0.672559		0.30455
10	2.38645657506403		0.198269		0.145249		0.745579		0.227431
11	2.57169539517539		0.198622		0.126216		0.821535		0.142267
12	2.75870117826714		0.198925		0.108069		0.897903		0.0519525
13	2.9471288020509		0.199158		0.0926932		0.965792		-0.0322945
14	3.13672349360904		0.199328		0.0804934		1.02207		-0.10526
$\infty$			0.2		0				

Table 5.3: Eighth moment ( $n = 4$ ) of the density when  $\Gamma = 4$ .

$N$	$\mathcal{M}_N$	$=$	$aN$	$+$	$b$	$+$	$c/\sqrt{N}$	$+$	$d/N$
			$a$		$b$		$c$		$d$
3	0.829151732377539								
4	1.13999055712937								
5	1.45889183119874								
6	1.78179400313294		0.330681		-0.23698		-0.0197034		0.256393
7	2.10668148567864		0.326556		-0.113869		-0.383961		0.55846
8	2.43308749295152		0.335184		-0.423364		0.621536		-0.358107
9	2.76069703430536		0.335669		-0.443708		0.693027		-0.428642
10	3.08920070504297		0.333408		-0.335393		0.285749		0.00150004
11	3.41837572685644		0.33287		-0.306358		0.169876		0.131419
12	3.74807370935471		0.333069		-0.318295		0.220112		0.0720079
$\infty$			1/3		-1/3				

Table 5.4: Fourth moment ( $n = 2$ ) of the density when  $\Gamma = 6$ .

$N$	$\mathcal{M}_N$	$=$	$aN$	$+$	$b$	$+$	$c/\sqrt{N}$	$+$	$d/N$
			$a$		$b$		$c$		$d$
3	0.63878932696137								
4	0.852317437834435								
5	1.07602482170551								
6	1.30481948267947		0.238952		-0.17935		-0.0015329		0.306508
7	1.5366992098612		0.241542		-0.256653		0.227188		0.116838
8	1.77130926929194		0.254997		-0.739333		1.79534		-1.31262
9	2.00817460819162		0.253514		-0.677215		1.57705		-1.09725
10	2.24678505124783		0.249869		-0.502619		0.920542		-0.403889
11	2.48676496530133		0.249348		-0.474543		0.8085		-0.278266
12	2.72785340315076		0.249797		-0.501405		0.921547		-0.411958
$\infty$			0.25		-0.5				

Table 5.5: Sixth moment ( $n = 3$ ) of the density when  $\Gamma = 6$ .

$N$	$\mathcal{M}_N$	$=$	$aN$	$+$	$b$	$+$	$c/\sqrt{N}$	$+$	$d/N$
			$a$		$b$		$c$		$d$
3	0.579848665870171								
4	0.732937257370685								
5	0.897778601877637								
6	1.06814688756782		0.178851		0.00107475		-0.227029		0.519884
7	1.24220552729734		0.18911		-0.305123		0.678947		-0.231411
8	1.41969193536163		0.204817		-0.8686		2.50959		-1.90014
9	1.60002816650081		0.201673		-0.736901		2.04679		-1.44354
10	1.78261079422029		0.198035		-0.562593		1.39137		-0.751319
11	1.96700538479197		0.198245		-0.573912		1.43654		-0.801964
12	2.15290808747152		0.199242		-0.633669		1.68802		-1.09938
$\infty$			0.2		$-2/3 \simeq -0.666667$				

Table 5.6: Eighth moment ( $n = 4$ ) of the density when  $\Gamma = 6$ .

$N$	$\mathcal{M}_N$	$=$	$aN$	$+$	$b$	$+$	$c/\sqrt{N}$	$+$	$d/N$
			$a$		$b$		$c$		$d$
3	0.691851631655437								
4	0.992192408359008								
5	1.30687171332381								
6	1.62802997072456		0.329038		-0.289875		-0.448497		0.76066
7	1.95072743432764		0.302864		0.491355		-2.7599		2.67751
$\infty$			1/3		-0.5				

Table 5.7: Fourth moment ( $n = 2$ ) of the density when  $\Gamma = 8$ .

$N$	$\mathcal{M}_N$	$=$	$aN$	$+$	$b$	$+$	$c/\sqrt{N}$	$+$	$d/N$
			$a$		$b$		$c$		$d$
3	0.462978101466508								
4	0.661530611956197								
5	0.877122153006864								
6	1.10026340398747		0.231106		-0.191724		-0.630085		0.97549
7	1.32616206590665		0.215749		0.266649		-1.98632		2.10017
$\infty$			0.25		-0.75				

Table 5.8: Sixth moment ( $n = 3$ ) of the density when  $\Gamma = 8$ .

$N$	$\mathcal{M}_N$	$=$	$aN$	$+$	$b$	$+$	$c/\sqrt{N}$	$+$	$d/N$
			$a$		$b$		$c$		$d$
3	0.359389450389747								
4	0.499919485631197								
5	0.656534473821437								
6	0.819712345574349		0.165419		0.0211561		-0.961295		1.19094
7	0.985867782818619		0.16514		0.0294668		-0.985884		1.21133
$\infty$			0.2		-1				

Table 5.9: Eighth moment ( $n = 4$ ) of the density when  $\Gamma = 8$ .

## 6 Conclusion

In this work we explored two applications to the 2dOCP of the expansion of powers of the Vandermonde determinant based on the formalism presented in section 2. The first is the study of the moments of the pair correlation function on the sphere. We showed that the second moment satisfies an exact relation for finite number of particles  $N$ , and we explored numerically the behavior of higher order moments. Also an approximation to these moments using a formalism based on the direct correlation function was proposed.

The second result is on the evaluation of the distribution function of the linear statistic  $\sum_{l=1}^N |\vec{r}_l|^{2n}$  in the soft disk geometry. The exact distribution tends to a Gaussian when  $N \rightarrow \infty$ . We were able to compute the  $O(1)$  correction to mean of this linear statistic for any  $n$ , and deduce it for a general smooth radial linear statistic. The result was checked explicitly at  $\Gamma = 2$ , for values of  $\Gamma$  close to 2, and numerically for  $\Gamma = 4$  and 6.

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## Appendix

In this appendix we present some properties of the functions

$$\mathcal{I}(k_1, k_2) = \iint_{0 \leq t_2 < t_1} e^{-t_1 - t_2} t_1^{k_1} t_2^{k_2} dt_1 dt_2, \quad (\text{A.1})$$

and

$$\mathcal{J}(k_1, k_2) = \int_0^\infty \int_0^\infty e^{-t_1 - t_2} t_1^{k_1} t_2^{k_2} \log(\max(t_1, t_2)) dt_1 dt_2, \quad (\text{A.2})$$

that appear in the expansion of the density around  $\Gamma = 2$ . First, it should be noticed that

$$\mathcal{I}(k_1, k_2) + \mathcal{I}(k_2, k_1) = k_1!k_2! . \quad (\text{A.3})$$

Doing an integration by parts, one obtains the recurrence relation

$$\mathcal{I}(k_1 + 1, k_2) - (k_1 + 1)\mathcal{I}(k_1, k_2) = 2^{-k_1 - k_2 - 2}(k_1 + k_2 + 1)! , \quad (\text{A.4})$$

and reiterating

$$\mathcal{I}(k_1 + n, k_2) - \frac{(k_1 + n)!}{k_1!} \mathcal{I}(k_1, k_2) = \sum_{\ell=0}^{n-1} 2^{-k_1 - k_2 - 2 - \ell} \frac{(k_1 + k_2 + 1 + \ell)!(k_1 + n)!}{(k_1 + \ell)!} . \quad (\text{A.5})$$

Similarly, for  $\mathcal{J}$  we have

$$\mathcal{J}(k_1 + n, k_2) - \frac{(k_1 + n)!}{k_1!} \mathcal{J}(k_1, k_2) = \sum_{\ell=0}^{n-1} \mathcal{I}(k_1 + \ell, k_2) \frac{(k_1 + n)!}{(k_1 + \ell + 1)!} . \quad (\text{A.6})$$

From (A.5), one can obtain an alternative expression for  $\mathcal{I}$  as a sum

$$\mathcal{I}(k_1, k_2) = \sum_{\ell=0}^{k_1} 2^{-k_2 - \ell - 1} \frac{(k_2 + \ell)!k_1!}{\ell!} . \quad (\text{A.7})$$

The asymptotic expansion of  $\mathcal{I}(k_1, k_2)$  for large arguments,  $k_1$  and  $k_2$  of order  $N \rightarrow \infty$ , can be obtained by the steepest descent method. The maximum of the integrand in (A.1) is for  $t_1 = k_1$  and  $t_2 = k_2$ . Therefore, the behavior of  $\mathcal{I}(k_1, k_2)$  will depend on whether this maximum is in the domain of integration  $0 \leq t_2 < t_1$  or not, i.e., if  $k_2 < k_1$  or not. If  $1 \ll k_2 < k_1$ , and  $|k_1 - k_2|/\sqrt{N} \gg 1$ , then the maximum of the integrand in (A.1) is deep inside the domain of integration, and a simple application of the steepest descent shows that  $\mathcal{I}(k_1, k_2) \sim k_1!k_2!$ . However for the calculations of section 5.3, the behavior of  $\mathcal{I}(k_1, k_2)$  when  $k_2 > k_1$  is needed. In this situation, the maximum of the integrand is outside the domain of integration. Nevertheless,  $\mathcal{I}(k_1, k_2)$  will give a significant contribution when the maximum of the integrand is “close” to the border, more precisely when  $|k_2 - k_1|$  is of order  $O(\sqrt{N})$ . The dominant contribution to the integral (A.1) will be given by the region consisting of a strip attached and parallel to the line  $t_1 = t_2$  of width of order  $\sqrt{N}$ . To be more specific, let us do the change of integration variables  $v_+ = t_1 + t_2$  and  $v_- = t_1 - t_2$  in (A.1) and write

$$\mathcal{I}(k_1, k_2) = \iint_{\mathcal{D}} dv_+ dv_- e^{-g(v_+, v_-)} \frac{dv_+ dv_-}{2^{k_1 + k_2 + 1}} , \quad (\text{A.8})$$

with

$$g(v_+, v_-) = v_+ - k_1 \log(v_+ + v_-) - k_2 \log(v_+ - v_-) , \quad (\text{A.9})$$

where  $\mathcal{D} = \{(t_1, t_2), 0 \leq t_2 < t_1\}$ .  $g$  has its maximum for  $v_+ = v_+^* = k_1 + k_2$  and  $v_- = v_-^* = k_1 - k_2$  (i.e.  $t_1 = k_1$  and  $t_2 = k_2$ ). Expanding  $g$  to the second order around its maximum we obtain

$$\mathcal{I}(k_1, k_2) \sim \frac{e^{-k_1 - k_2} k_1^{k_1} k_2^{k_2}}{2} \iint_{\mathcal{D}} e^{-\frac{k_1 + k_2}{8k_1 k_2} [(v_+ - v_+^*)^2 + (v_- - v_-^*)^2] - \frac{k_2 - k_1}{4k_1 k_2} (v_+ - v_+^*)(v_- - v_-^*)} dv_+ dv_- . \quad (\text{A.10})$$

Although the domain of integration  $\mathcal{D}$  cannot be simply expressed in terms of the variables  $v_+$  and  $v_-$ , due to fast Gaussian decay of the integrand in (A.10), the dominant contribution is indeed obtained by the strip along the line  $t_1 = t_2$  mentioned above. With this in mind, it is clear that one can extend the domain of integration for  $v_+$  to  $]-\infty, +\infty[$  and for  $v_-$  to  $[0, +\infty[$ , up to exponentially small corrections. Then, performing first the integral over  $v_+$ , we obtain

$$\mathcal{I}(k_1, k_2) \sim e^{-k_1-k_2} k_1^{k_1} k_2^{k_2} \sqrt{\frac{2\pi k_1 k_2}{k_1 + k_2}} \int_0^{+\infty} e^{-\frac{(v_- - v_-^*)^2}{2(k_1 + k_2)}} dv_- . \quad (\text{A.11})$$

Performing now the integral over  $v_-$  gives

$$\mathcal{I}(k_1, k_2) \sim e^{-k_1-k_2} k_1^{k_1} k_2^{k_2} \pi \sqrt{k_1 k_2} \operatorname{erfc} \left( \frac{k_2 - k_1}{\sqrt{2(k_1 + k_2)}} \right) , \quad (\text{A.12})$$

where  $\operatorname{erfc}(x) = 1 - (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$ , is the complementary error function. Recalling Stirling's formula for the factorial, (A.12) can be written as

$$\frac{\mathcal{I}(k_1, k_2)}{k_1! k_2!} \sim \frac{1}{2} \operatorname{erfc} \left( \frac{k_2 - k_1}{\sqrt{2(k_1 + k_2)}} \right) . \quad (\text{A.13})$$

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